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Common fixed point theorems under rational contractions in complex valued extended b-metric spaces

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Abstract

In this paper, we discuss the existence and uniqueness of fixed point and common fixed point theorems in complex valued extended b-metric space for a pair of mappings satisfying some rational contraction conditions which generalize and unify some well known results in the literature.

Keywords: Complex valued extended b-metric space, Rational contraction, Common fixed point.
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1. Introduction

The Fixed point theory is a well known research field in mathematical sciences. Fixed point technique is an important tool in the area of the non-linear analysis. The Banach contraction mapping principle [3] plays a vital role in fixed point theory. In 1969, Nadler [13] developed the fixed point theorems for multi-valued mappings. Huange and Zhange [9] initiated the concept of cone metric space as a generalization of metric spaces. The well-known fixed point results involving rational contractions could not be extended in cone metric spaces. To rectify this restriction, Azam et al. [1] developed the concept of complex valued metric spaces and introduced sufficient conditions involving rational expressions. In 1989 Bakhtin [2] presented a new space called b-metric space which is the generalization of metric space. Czerwik [7] extended the Banach principle in b-metric space. Many researchers proved fixed point theorems on single valued and multi valued mapping in b-metric space

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[5] [10]. Rao et al. [14] introduced complex valued b-metric space, continuously Mukheimer [12], A. K. Dubey [8], Dayana [15], Carmel [6] verified the existence of some common fixed point theorems in complex valued b-metric space. In 2017, Kamran et al [11] introduced extended b- metric space and Naimat Ullah and et al [16] initiated the concept of complex valued extended b-metric spaces. In this paper, we prove fixed point theorems in complex valued extended b-metric space by using rational contractions.

2. Preliminaries

Definition 2.1. *[1] Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:*
$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Thus $z_1 \leq z_2$ if one of the following holds:

1. $Re(z_1) = Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;
2. $Re(z_1) < Re(z_2)$ *and* $Im(z_1) = Im(z_2)$;
3. $Re(z_1) = Re(z_2)$ *and* $Im(z_1) < Im(z_2)$;
4. $Re(z_1) < Re(z_2)$ *and* $Im(z_1) < Im(z_2)$;

We will write$z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3), and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.
It follows that

(i) $0 \preceq z_1 \precnsim z_2$ *implies* $|z_1| < |z_2|$;
(ii) $z_1 \preceq z_2$ *and* $z_2 \prec z_3$ *imply* $z_1 \prec z_3$;
(iii) $0 \preceq z_1 \preceq z_2$ *implies* $|z_1| \leq |z_2|$;
(iv) *if* $a, b \in \mathbb{R}$, $0 \leq a \leq b$ *and* $z_1 \preceq z_2$ *then* $az_1 \prec bz_2$ *for all* $z_1, z_2 \in \mathbb{C}$.

Definition 2.2. *[1] Let W be a non-empty set. A function $d_{cv} : W \times W \to \mathbb{C}$ is called a complex valued metric on W, if for all $l, m, n \in W$, the following conditions are satisfied:*

(i) $0 \preceq d_{cv}(l, m)$ *and* $d_{cv}(l, m) = 0$ *if and only if* $l = m$;
(ii) $d_{cv}(l, m) = d_{cv}(m, l)$;
(iii) $d_{cv}(l, m) \preceq d_{cv}(l, n) + d_{cv}(n, m)$.

Then the pair (W, d_{cv}) is called a complex valued metric space.

Example 2.3. *[1] Let $W = [0, 1]$ and $l, m \in W$. Define $d_{cv} : W \times W \to \mathbb{C}$ by*

$$d_{cv}(l, m) = \begin{cases} 0 & \text{if } l = m \\ \dfrac{i}{2} & \text{if } l \neq m \end{cases} \tag{2.1}$$

Then d_{cv} is a complex valued metric and hence (W, d_{cv}) is a complex valued metric space.

Definition 2.4. *[1] Let (W, d_{cv}) be a complex valued metric space.*

(i) We say that a point $l \in W$ is an interior point of a set $M \subseteq W$, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(l, r) = m \in W : d_{cv}(m, l) \prec r,$$

(ii) *We say that a point* $l \in W$ *is a limit point of a set* $M \subseteq W$*, whenever for every* $0 \prec r \in \mathbb{C}$ *such that*

$$B(l, r) \cap M - l \neq \emptyset.$$

Definition 2.5. *[2][7]Let* W *be a non-empty set and* $s \geq 1$ *be a given real number. A function* $d_b : W \times W \to [0, \infty)$ *is called b-metric on* W *if for all* $l, m, n \in W$*, the following conditions are satisfied:*

(b1) $d_b(l, m) = 0$ *if and only if* $l = m$*;*
(b2) $d_b(l, m) = d_b(m, l)$*;*
(b3) $d_b(l, m) \leq s[d_b(l, n) + d_b(n, m)]$.

Then the pair (W, d_b) *is called a b-metric space.*

Example 2.6. *[4] Let* $W = L_p[0, 1]$ *be the space of all real functions* $l(t), t \in [0, 1]$ *such that* $\int_0^1 |l(t)|^p < \infty$ *with* $0 < p < 1$*. Define* $d_b : W \times W \to \mathbb{R}^+$ *as:*

$$d_b(l, m) = \left(\int_0^1 |l(t) - m(t)|^p dt \right)^{\frac{1}{p}}$$

then (W, d_b) *is b-metric space with coefficient* $s = 2^{\frac{1}{p}}$.

Definition 2.7. *[14] Let* W *be a non-empty set and let* $s \geq 1$ *be a given real number. A function* $d_{cvb} : W \times W \to \mathbb{C}$ *is called a complex valued b-metric on* W *if for all* $l, m, n \in W$*, the following conditions are satisfied:*

(i) $0 \preceq d_{cvb}(l, m)$ *and* $d_{cvb}(l, m) = 0$ *if and only if* $l = m$*;*
(ii) $d_{cvb}(l, m) = d_{cvb}(m, l)$*;*
(iii) $d_{cvb}(l, m) \preceq s[d_{cvb}(l, n) + d_{cvb}(n, m)]$.

Then the pair (W, d_{cvb}) *is called a complex valued b-metric space.*

Example 2.8. *[14] If* $W = [0, 1]$*, define the mapping* $d_{cvb} : W \times W \to \mathbb{C}$ *by*

$$(l, m) = |l - m|^2 + i|l - m|^2$$

for all $l, m \in W$*. Then* (W, d_{cvb}) *is a complex valued b-metric space with* $s = 2$.

Definition 2.9. *[11] Let* W *be a non-empty set and* $\lambda : W \times W \to [1, \infty)$ *be a function. Then* $d_\lambda : W \times W \to [0, \infty)$ *is called an extended b-metric if for all* $l, m, n \in W$ *it satisfies:*

(i) $d_\lambda(l, m) = 0$ *if and only if* $l = m$*;*
(ii) $d_\lambda(l, m) = d_\lambda(m, l)$*;*
(iii) $d_\lambda(l, n) \leq \lambda(l, n)[d_\lambda(l, m) + d_\lambda(m, n)]$.

Then the pair (W, d_λ) *is called an extended b-metric space.*

Example 2.10. *Let $W = 1, 2, 3$. Define $\lambda : W \times W \to \mathbb{R}^+$ and $d_\lambda : W \times W \to \mathbb{R}^+$ as:*

$$\begin{aligned}
&\lambda(l, m) = 1 + l + m \\
&d_\lambda(1, 1) = d_\lambda(2, 2) = d_\lambda(3, 3) = 0 \\
&d_\lambda(1, 2) = d_\lambda(2, 1) = 80, d_\lambda(1, 3) = d_\lambda(3, 1) = 1000 \\
&d_\lambda(2, 3) = d_\lambda(3, 2) = 600
\end{aligned}$$

then (W, d_λ) is an extended b-metric space.

Definition 2.11. *[11] Let (W, d_λ) be an extended b-metric space.*

(i) A sequence $\{l_n\}$ in W is said to converge to $l \in W$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_n, l) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} l_n = l$.

(ii) A sequence $\{l_n\}$ in W is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_m, l_n) < \epsilon$, for all $m, n \geq N$.

(iii) If every Cauchy sequence in W is convergent, then (W, d_λ) is said to be a complete extended b-metric space.

Lemma 2.12. *[11] Let (W, d_λ) be an extended b-metric space. If d_λ is continuous, then every convergent sequence has a unique limit.*

Definition 2.13. *[16] Let W be a non-empty set and $\theta : W \times W \to [1, \infty)$ be a function. Then $d_\theta : W \times W \to \mathbb{C}$ is known as a complex valued b-metric space if the following conditions are satisfied for all $l, m, n \in W$:*

(i) $0 \preceq d_\theta(l, m)$ and $d_\theta(l, m) = 0$ if and only if $l = m$;

(ii) $d_\theta(l, m) = d_\theta(m, l)$;

(iii) $d_\theta(l, n) \preceq \theta(l, n)[d_\theta(l, m) + d_\theta(m, n)]$.

Then the pair (W, d_θ) is called a complex valued extended b-metric space.

Example 2.14. *If W be a non-empty set and $\theta : W \times W \to [1, \infty]$ be defined as:*

$$\theta(l, m) = \frac{1 + l + m}{l + m}$$

further, Let

(i) $d_\theta(l, m) = \dfrac{i}{lm}$ for all $l, m \in (0, 1]$;

(ii) $d_\theta(l, m) = 0 \iff l = m$ for all $l, m \in [0, 1]$;

(iii) $d_\theta(l, 0) = d_\theta(0, l) = \dfrac{i}{l}$ for all $l \in (0, 1]$.

Then the pair (W, d_θ) is known as a complex valued extended b-metric space.

Example 2.15. *Let $W = [0, \infty)$. $\theta : W \times W \to [1, \infty)$ be a function defined by $\theta(l, m) = 1 + l + m$ and $d_\theta : W \times W \to \mathbb{C}$ be given as*

$$d_\theta(l, m) = \left\{ \begin{array}{ll} 0 & \text{if } l = m \\ i & \text{if } l \neq m \end{array} \right\}$$

Then (W, d_θ) is a complex valued extended b - metric space.

3. Main results

Theorem 3.1. *Let (W, d_θ) be a complete complex valued extended b-metric space; let $\theta : W \times W \to [1,\infty)$ and let U, V be self-mappings from W into itself satisfy the following inequality:*

$$d_\theta(Ul, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, Ul)d_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m)} \tag{3.1}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) \neq 0$ where μ_1 and μ_2 are non negative reals with $\mu_1 + \mu_2\theta(l_1, l_2) < 1$, $\zeta = \mu_1 + \mu_2\theta(l_1, l_2)$ where $\zeta \in [0,\infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) = 0$. Then U and V have a unique common fixed point in W.

Proof . For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = V l_{2n+1} \quad \forall n \geq 0 \tag{3.2}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence.
Let $l = l_0$, $m = l_1$ in (3.1)

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, V l_1) \\ &\preceq \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, Ul_0)d_\theta(l_1, V l_1)}{d_\theta(l_0, V l_1) + d_\theta(l_1, Ul_0) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)}. \end{aligned}$$

Then

$$|d_\theta(l_1, l_2)| = \mu_1|d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}$$

Using triangular inequality

$$d_\theta(l_1, l_2) \leq \theta(l_1, l_2)[d_\theta(l_1, l_0) + d_\theta(l_0, l_2)]$$

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1|d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_1, l_2)|}|\theta(l_1, l_2)| \\ &= (\mu_1 + \mu_2\theta(l_1, l_2))|d_\theta(l_0, l_1)| \\ |d_\theta(l_1, l_2)| &\leq (\mu_1 + \mu_2\theta(l_1, l_2))|d_\theta(l_0, l_1)|. \end{aligned}$$

Since $|d_\theta(l_1, l_2)| < 1 + |d_\theta(l_1, l_2)|$,

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \zeta|d_\theta(l_0, l_1)| \\ |d_\theta(l_2, l_3)| &\leq \zeta^2|d_\theta(l_0, l_1)| \\ |d_\theta(l_3, l_4)| &\leq \zeta^3|d_\theta(l_0, l_1)| \\ &\vdots \\ |d_\theta(l_n, l_{n+1})| &\leq \zeta^n|d_\theta(l_0, l_1)| \end{aligned}$$

Now, by triangular inequality, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$d_\theta(l_n, l_m) \preceq \theta(l_n, l_m)\zeta^n d_\theta(l_0, l_1) + \theta(l_n, l_m)\theta(l_{n+1}, l_m)\zeta^{n+1} d_\theta(l_0, l_1) \dots + \theta(l_n, l_m)\theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m)\theta(l_{m-1}, l_m)\zeta^{m-1} d_\theta(l_0, l_1).$$

Then

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[\theta(l_n, l_m)\zeta^n + \theta(l_n, l_m)\theta(l_{n+1}, l_m)\zeta^{n+1} \dots + \theta(l_n, l_m)\theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m)\theta(l_{m-1}, l_m)\zeta^{m-1}]$$

Since, $\lim_{n,m\to\infty} \theta(l_n, l_m)\zeta < 1$, so the series $\sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^{n} \theta(l_i, l_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^{n} \theta(l_i, l_m),\ S_n = \sum_{j=1}^{n} \zeta^j \prod_{i=1}^{j} \theta(l_i, l_m)$$

Thus, for $m > n$, the above can be written as

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[S_{m-1} - S_n] \text{ and}$$

$$|d_\theta(l_n, l_m)| \leq |d_\theta(l_0, l_1)|[S_{m-1} - S_n]$$

Letting $n \to \infty$, we obtain

$$|d_\theta(l_n, l_m)| \to 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W. Since W is complete there exists some $t \in W$ such that $l_n \to t$ as $n \to \infty$.
Assume not, then there exits $z \in W$ such that

$$|d_\theta(t, Ut)| = |z| > 0. \tag{3.3}$$

So by using the triangular inequality and (3.1), we have

$$\begin{aligned}
z &= d_\theta(t, Ut) \\
&\preceq \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)d_\theta(l_{2n+2}, Ut) \\
&= \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)d_\theta(Vl_{2n+1}, Ut) \\
&\preceq \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)\mu_1 d_\theta(t, l_{2n+1}) \\
&\quad + \theta(t, Ut)\mu_2 \frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, Vl_{2n+1})}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})} \\
&= \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)\mu_1 d_\theta(t, l_{2n+1}) \\
&\quad + \theta(t, Ut)\mu_2 \frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, l_{2n+2})}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})}
\end{aligned}$$

$$|z| = |d_\theta(t, Ut)| \leq \theta(t, Ut)|d_\theta(t, l_{2n+2})| + \theta(t, Ut)\mu_1|d_\theta(t, l_{2n+1})| + \theta(t, Ut)\mu_2 \frac{|d_\theta(t, Ut)||d_\theta(l_{2n+1}, l_{2n+2})|}{|d_\theta(t, l_{2n+2})| + |d_\theta(l_{2n+1}, Ut)| + |d_\theta(t, l_{2n+1})|}$$

As $n \to \infty$, we obtain that $|z| = |d_\theta(t, Ut)| \leq 0$, a contradiction.
Thus, $|z| = 0$.
Hence, $Ut = t$. Similarly, we obtain $Vt = t$.
Now, we show that U and V have a unique common fixed point. To prove this,assume that $t^{'} \neq t$ is another common fixed point of U and V. Then

$$\begin{aligned} d_\theta(t, t') &= d_\theta(Ut, Vt') \\ &\preceq \mu_1 d_\theta(t, t') + \mu_2 \frac{d_\theta(t, Ut) d_\theta(t', Vt')}{d_\theta(t, Vt') + d_\theta(t', Ut) + d_\theta(t, t')} \end{aligned}$$

Then,

$$\begin{aligned} |d_\theta(t, t')| &\leq \mu_1 |d_\theta(t, t')| + \mu_2 \frac{|d_\theta(t, Ut)||d_\theta(t', Vt')|}{|d_\theta(t, Vt')| + |d_\theta(t', Ut)| + |d_\theta(t, t')|} \\ |d_\theta(t, t')| &\leq \mu_1 |d_\theta(t, t')|, \end{aligned}$$

which is a contradiction.
Hence, $t = t'$, which shows the uniqueness of common fixed point in W.

Now, we consider the second case:

$d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) = 0$.
$l = l_{2n}$ and $m = l_{2n+1}$.
$d_\theta(l_{2n}, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ul_{2n}) + d_\theta(l_{2n}, l_{2n+1}) = 0$
$d_\theta(Ul_{2n}, Vl_{2n+1}) = 0$
So that $l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}$
Thus,we have $l_{2n+1} = Ul_{2n} = l_{2n}$, so there exist E_1 and f_1 such that
$E_1 = Uf_1 = f_1$ where $E_1 = l_{2n+1}$ and $f_1 = l_{2n}$.
Using foregoing arguments, we show that there exist E_2 and f_2
such that $E_2 = Vf_2 = f_2$ where $E_2 = l_{2n+2}$ and $f_2 = l_{2n+1}$.
As, $d_\theta(f_1, Vf_2) + d_\theta(f_2, Uf_1) + d_\theta(f_1, f_2) = 0$ which implies
$d_\theta(Uf_1, Vf_2) = 0$. $E_1 = Uf_1 = Vf_2 = E_2$.
Thus we obtain $E_1 = Uf_1 = UE_1$.
Similarly,we have $E_2 = VE_2$.
As $E_1 = E_2 \Rightarrow UE_1 = VE_1 = E_1$,
Hence $E_1 = E_2$ is common fixed point of U and V.
For uniqueness of common fixed point, assume that E_1' in W is another common fixed point of U and V. Then we have $UE_1' = VE_1' = E_1'$
As $d_\theta(E_1, VE_1') + d_\theta(E_1', UE_1) + d_\theta(E_1, E_2') = 0$,
therefore $d_\theta(E_1, E_1') = d_\theta(UE_1, VE_1') = 0$
This implies that $E_1 = E_1'$.
This completes the proof of the theorem. □

Corollary 3.2. *Let (W, d_θ) be a complete complex valued extended b-metric space; let $\theta : W \times W \to [1, \infty)$ and let $V : W \to W$ be a mapping satisfying:*

$$d_\theta(Vl, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, Vl) d_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m)} \tag{3.4}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) \neq 0$ where μ_1 and μ_2 are non negative reals with $\mu_1 + \mu_2\theta(l_1, l_2) < 1$, $\zeta = \mu_1 + \mu_2\theta(l_1, l_2)$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Vl, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) = 0$. Then V has a unique fixed point in W.

Proof . By using theorem 3.1 with $U = V$, we can prove this result. □

Corollary 3.3. *Let (W, d_θ) be a complete complex valued extended b-metric space; let $\theta : W \times W \to [1, \infty)$ and let $V : W \to W$ be a mapping satisfying (for some fixed n),*

$$d_\theta(V^n l, V^n m) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, V^n l) d_\theta(m, V^n m)}{d_\theta(l, V^n m) + d_\theta(m, V^n l) + d_\theta(l, m)} \tag{3.5}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, V^n m) + d_\theta(m, V^n l) + d_\theta(l, m) \neq 0$ where μ_1 and μ_2 are non negative reals with $\mu_1 + \mu_2\theta(l_1, l_2) < 1$, $\zeta = \mu_1 + \mu_2\theta(l_1, l_2)$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or
$d_\theta(V^n l, V^n m) = 0$ if
$d_\theta(l, V^n m) + d_\theta(m, V^n l) + d_\theta(l, m) = 0$. Then V has a unique fixed point in W.

Proof . By using corollary 3.2 with $V = V^n$, we can prove this result. □

Example 3.4. *Let $W = [0, \infty)$. Define $\theta : W \times W \to [1, \infty)$ by*

$$\theta(l, m) = \frac{2 + l + m}{1 + l + m} \text{ for all} l, m \in W,$$

and $d_\theta : W \times W \to \mathbb{C}$ by

$$d_\theta(l, m) = |l - m|^2 + i|l - m|^2 \text{ for all} l, m \in W$$

Then (W, d_θ) is a complex valued extended b - metric space with $s = 2$. Consider the mappings $U, V : W \to W$ defined by

$$Ul = \begin{cases} \left[0, \frac{l}{5}\right], & \text{if } l \in [0, 1] \\ [l, 3l], & \text{otherwise.} \end{cases}$$

$$Vl = \begin{cases} \left[0, \frac{l}{10}\right], & \text{if } l \in [0, 1] \\ [3l, 7l], & \text{otherwise.} \end{cases}$$

If $l = m = 0$, conditions of Theorem 3.1 hold trivially. Suppose l and m are non zero with $l < m$. Then

$$d_\theta(l, Ul) = |l - \frac{l}{5}|^2 + i|l - \frac{l}{5}|^2,$$
$$d_\theta(m, Vm) = |m - \frac{m}{10}|^2 + i|m - \frac{m}{10}|^2,$$
$$d_\theta(m, Ul) = |m - \frac{l}{5}|^2 + i|m - \frac{l}{5}|^2,$$
$$d_\theta(l, Vm) = |l - \frac{m}{10}|^2 + i|l - \frac{m}{10}|^2,$$
$$s(Ul, Vm) = s\left(|\frac{l}{5} - \frac{m}{10}|^2 + i|\frac{l}{5} - \frac{m}{10}|^2\right).$$

By taking $\mu_1 = \frac{1}{2}$ and $\mu_2 = 0$, it can be verified that all the conditions of Theorem 3.1 are satisfied. Hence 0 is a common fixed point of U and V.

Theorem 3.5. *Let (W, d_θ) be a complete complex valued extended b-metric space; let and let U, V be self-mappings from W into itself satisfy the following inequality,*

$$d_\theta(Ul, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2[d_\theta(l, Ul) + d_\theta(m, Vm)] + \mu_3 \frac{[d_\theta^2(l, Vm) + d_\theta^2(m, Ul)]}{d_\theta(l, Vm) + d_\theta(m, Ul)} \tag{3.6}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Ul) \neq 0$ where μ_1, μ_2 and μ_3 are non negative reals with $\mu_1 + 2\mu_2 + 2\theta(l_0, l_2)\mu_3 < 1$, $\zeta(1 - \mu_2 - \mu_3\theta(l_0, l_2)) = (\mu_1 + \mu_2 + \mu_3\theta(l_0, l_2))$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) = 0$. Then U and V have a unique common fixed point in W.

Proof . For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = Vl_{2n} \quad \forall n \geq 0 \tag{3.7}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence.
Let $l = l_0$, $m = l_1$ in(3.6).

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, Vl_1) \\ &\preceq \mu_1 d_\theta(l_0, l_1) + \mu_2[d_\theta(l_0, Ul_0) + d_\theta(l_1, Vl_1)] + \mu_3 \frac{[d_\theta^2(l_0, Vl_1) + d_\theta^2(l_1, Ul_0)]}{d_\theta(l_0, Vl_1) + d_\theta(l_1, Ul_0)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] + \mu_3 \frac{[d_\theta^2(l_0, l_2) + d_\theta^2(l_1, l_1)]}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] + \mu_3 \frac{[d_\theta^2(l_0, l_2)]}{d_\theta(l_0, l_2)} \end{aligned}$$

Then

$$|d_\theta(l_1, l_2)| \leq \mu_1|d_\theta(l_0, l_1)| + \mu_2[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] + \mu_3 \frac{[|d_\theta^2(l_0, l_2)|]}{|d_\theta(l_0, l_2)|}$$
$$|d_\theta(l_1, l_2)| \leq \mu_1|d_\theta(l_0, l_1)| + \mu_2[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] + \mu_3|d_\theta(l_0, l_2)|$$

Using triangular inequality, we have

$$d_\theta(l_0, l_2)| \leq \theta(l_0, l_2)[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)]$$

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1|d_\theta(l_0, l_1)| + \mu_2[|d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] + \mu_3\theta(l_0, l_2)[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ &= (\mu_1 + \mu_2 + \mu_3\theta(l_0, l_2))|d_\theta(l_0, l_1)| + (\mu_2 + \mu_3\theta(l_0, l_2))|d_\theta(l_1, l_2)| \\ |d_\theta(l_1, l_2)| &= \frac{(\mu_1 + \mu_2 + \mu_3\theta(l_0, l_2))}{(1 - \mu_2 - \mu_3\theta(l_0, l_2))}|d_\theta(l_0, l_1)| \end{aligned}$$

Then, we obtain

$$|d_\theta(l_1, l_2)| \leq \zeta|d_\theta(l_0, l_1)|$$

Similarly,

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \zeta|d_\theta(l_0, l_1)| \\ |d_\theta(l_2, l_3)| &\leq \zeta^2|d_\theta(l_0, l_1)| \\ |d_\theta(l_3, l_4)| &\leq \zeta^3|d_\theta(l_0, l_1)| \\ &\vdots \\ |d_\theta(l_n, l_{n+1})| &\leq \zeta^n|d_\theta(l_0, l_1)| \end{aligned}$$

Now, by triangular inequality, for any $m > n$, $m, n \in \mathbb{N}$ we have

$$d_\theta(l_n, l_m) \preceq \theta(l_n, l_m)\zeta^n d_\theta(l_0, l_1) + \theta(l_n, l_m)\theta(l_{n+1}, l_m)\zeta^{n+1} d_\theta(l_0, l_1) \ldots + \theta(l_n, l_m)\theta(l_{n+1}, l_m) \ldots \theta(l_{m-2}, l_m)\theta(l_{m-1}, l_m)\zeta^{m-1} d_\theta(l_0, l_1)$$

Then

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[\theta(l_n, l_m)\zeta^n + \theta(l_n, l_m)\theta(l_{n+1}, l_m)\zeta^{n+1} \ldots + \theta(l_n, l_m)\theta(l_{n+1}, l_m) \ldots \theta(l_{m-2}, l_m)\theta(l_{m-1}, l_m)\zeta^{m-1}].$$

Since, $\lim\limits_{n,m\to\infty} \theta(l_n, l_m)\zeta < 1$, series $\sum\limits_{n=1}^{\infty} \zeta^n \prod\limits_{i=1}^{n} \theta(l_i, l_m)$ converges by ratio test for each $m \in \mathbb{N}$.
Let

$$S = \sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^{n} \theta(l_i, l_m),\ S_n = \sum_{j=1}^{n} \zeta^j \prod_{i=1}^{j} \theta(l_i, l_m)$$

Thus, for $m > n$, the above expression can be written as

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[S_{m-1} - S_n]$$

and

$$|d_\theta(l_n, l_m)| \leq |d_\theta(l_0, l_1)|[S_{m-1} - S_n]$$

Letting $n \to \infty$, we get

$$|d_\theta(l_n, l_m)| \to 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W. Since W is complete there exists some $t \in W$ such that $\{l_n\} \to t$ as $n \to \infty$.
Assume not, then there exits $z \in W$ such that

$$|d_\theta(t, Ut)| = |z| > 0. \tag{3.8}$$

Using the triangular inequality, we have

$$\begin{aligned}
z &= d_\theta(t, Ut)\\
&\preceq \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)d_\theta(l_{2n+2}, Ut)\\
&= \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)d_\theta(Vl_{2n+1}, Ut)\\
&\preceq \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)\mu_1 d_\theta(t, l_{2n+1}) + \theta(t, Ut)\mu_2[d_\theta(t, Ut) + d_\theta(l_{2n+1}, Vl_{2n+1})]\\
&\qquad + \theta(t, Ut)\mu_3 \frac{[d_\theta^2(t, Vl_{2n+1}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut)}\\
&= \theta(t, Ut)d_\theta(t, l_{2n+2}) + \theta(t, Ut)\mu_1 d_\theta(t, l_{2n+1}) + \theta(t, Ut)\mu_2[d_\theta(t, Ut) + d_\theta(l_{2n+1}, l_{2n+2})]\\
&\qquad + \theta(t, Ut)\mu_3 \frac{[d_\theta^2(t, l_{2n+2}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut)}
\end{aligned}$$

$$|z| = |d_\theta(t, Ut)| \leq |\theta(t, Ut)| \Bigg(|d_\theta(t, l_{2n+2})| + \mu_1 |d_\theta(t, l_{2n+1}) + \mu_2 \big[|d_\theta(t, Ut)| + |d_\theta(l_{2n+1}, l_{2n+2})|\big] + \mu_3 \frac{[|d_\theta^2(t, l_{2n+2})| + |d_\theta^2(l_{2n+1}, Ut)|]}{|d_\theta(t, l_{2n+2})| + |d_\theta(l_{2n+1}, Ut)|} \Bigg)$$

As $n \to \infty$, we obtain that $|z| = |d_\theta(t, Ut)| \leq 0$, a contradiction.
Thus, $|z| = 0$.
Hence, $Ut = t$. Similarly, we obtain $Vt = t$.
Now, we show that U and V have a unique common fixed point. To prove this,assume that $t^{'} \neq t$ is another common fixed point of U and V. Then

$$\begin{aligned} d_\theta(t, t') &= d_\theta(Ut, Vt') \\ &\preceq \mu_1 d_\theta(t, t') + \mu_2[d_\theta(t, Ut) + d_\theta(t', Vt')] + \mu_3 \frac{[d_\theta^2(t, Vt') + d_\theta^2(t', Ut)]}{d_\theta(t, Vt') + d_\theta(t', Ut)} \end{aligned}$$

Then,

$$\begin{aligned} |d_\theta(t, t')| &\leq \mu_1 |d_\theta(t, t')| + \mu_2[|d_\theta(t, Ut)| + |d_\theta(t', Vt')|] + \mu_3 \frac{[|d_\theta^2(t, Vt')| + |d_\theta^2(t', Ut)|]}{|d_\theta(t, Vt')| + |d_\theta(t', Ut)|} \\ |d_\theta(t, t')| &\leq (\mu_1 + \mu_3)|d_\theta(t, t')|, \end{aligned}$$

which is a contradiction. Hence $t = t'$ which shows the uniqueness of common fixed point in W.
For the second case, $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) = 0$,the proof of unique common fixed point can be completed in the line of Theorem 3.1.
This completes the proof of the theorem. □

Corollary 3.6. *Let (W, d_θ) be a complete complex valued extended b-metric space; let $\theta : W \times W \to [1, \infty)$ and let V be self-mapping from W into itself satisfy the following inequality,*

$$d_\theta(Vl, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2[d_\theta(l, Vl) + d_\theta(m, Vm)] + \mu_3 \frac{[d_\theta^2(l, Vm) + d_\theta^2(m, Vl)]}{d_\theta(l, Vm) + d_\theta(m, Vl)}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Vl) \neq 0$ where μ_1, μ_2 and μ_3 are non negative reals with $\mu_1 + 2\mu_2 + 2\theta(l_0, l_2)\mu_3 < 1$, $\zeta(1 - \mu_2 - \mu_3\theta(l_0, l_2)) = (\mu_1 + \mu_2 + \mu_3\theta(l_0, l_2))$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Vl, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Vl) = 0$ Then V has a unique fixed point in W.

Proof . By using the theorem 3.5 with $U = V$, we can prove this result. □

Corollary 3.7. *Let (W, d_θ) be a complete complex valued extended b-metric space; let $\theta : W \times W \to [1, \infty)$ and let $V : W \to W$ be a mapping satisfying (for some fixed n)*

$$d_\theta(V^n l, V^n m) \preceq \mu_1 d_\theta(l, m) + \mu_2[d_\theta(l, V^n l) + d_\theta(m, V^n m)] + \mu_3 \frac{[d_\theta^2(l, V^n m) + d_\theta^2(m, V^n l)]}{d_\theta(l, V^n m) + d_\theta(m, V^n l)}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, V^n m) + d_\theta(m, V^n l) \neq 0$ where μ_1, μ_2 and μ_3 are non negative reals with $\mu_1 + 2\mu_2 + 2\theta(l_0, l_2)\mu_3 < 1$, $\zeta(1 - \mu_2 - \mu_3\theta(l_0, l_2)) = (\mu_1 + \mu_2 + \mu_3\theta(l_0, l_2))$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(V^n l, V^n m) = 0$ if $d_\theta(l, V^n m) + d_\theta(m, V^n l) = 0$ Then V has a unique fixed point in W.

Proof . By using the corollary 3.6 with $V = V^n$, we can prove this result. □

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