Existence and convergence of best proximity points in G-metric spaces

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Outline

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2. Introduction
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Abstract

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In this paper, we introduce the new concept of cyclic G-contraction mapping and also we prove existence and convergence of best proximity point theorems in G-metric spaces.

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$$\begin{aligned} dist(A,B) &= \inf\{d(a,b) : a \in A, b \in B\} \\ A_0 &= \{a \in A : d(a,b) = dist(A,B) \text{ for some } \ b \in B\} \\ B_0 &= \{b \in B : d(a,b) = dist(A,B) \text{ for some } \ a \in A\} \end{aligned}$$

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G-metric space

Let X be a non-empty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function to satisfy the following axioms:

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Let (X, G) and (X', G') be $G-$metric spaces and let $f : (X, G) \to (X', G')$ be function, then f is said to be $G-$continuous at a point $a \in X$; if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(a, x, y) < \delta$ implies that

$$G'\big(f(a), f(x), f(y)\big) < \epsilon.$$

A function f is $G-$continuous on X if and only if it is $G-$continuous at all $a \in X$.

Preliminaries

Proposition

Let (X, G) and (X', G') be $G-$metric spaces, then a function $f : X \to X'$ is $G-$continuous at a point $x \in X$ if and only if it is $G-$sequentially continuous at x, that is, whenever $\{x_n\}$ is $G-$convergent to x, $\left\{f(x_n)\right\}$ is $G-$convergent to $f(x)$.

Preliminaries

Definition

Let (X, G) be a $G-$metric space, let $\{x_n\}$ be a sequence of points of X; therefore, it is said that $\{x_n\}$ is $G-$convergent to x if

$$\lim_{n,n\to\infty} G(x, x_n, x_m) = 0;$$

that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. One call x, the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

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Let (X, G) be a $G-$metric space. A sequence $\{x_n\}$ is called a $G-$Cauchy if, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_m, x_n, x_l) < \epsilon$, for all $n, m, l \geq N$; that is, $G(x_m, x_n, x_l) \to 0$ as $n, m, l \to \infty$.

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Proposition

Let (X, G) be a $G-$metric space. Then the following are equivalent;

1. The sequence $\{x_n\}$ is $G-$Cauchy
2. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$

Preliminaries

Definition

A $G-$metric space (X, G) is $G-$complete if every G-Cauchy sequence in (X, G) is $G-$convergent.

Preliminaries

Example

Let $(\mathbb{R}, d)$ be the usual metric space. Define G_a by $G_a(x, y, z) = d(x, y) + d(y, z) + d(x, z)$ for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_a)$ is a $G-$metric space.

Preliminaries

Best proximity point in G-metric spaces

Let (X, G) be a G-metric space and let A, B, and C be non-empty subsets of X. A mapping $T : A \cup B \cup C \to A \cup B \cup C$ is such that $T(A) \subset B, T(B) \subset C$, and $T(C) \subset A$. We call an element $x \in A \cup B \cup C$ a best proximity point (with respect to T) if $G(x, Tx, T^2x) = G(A, B, C)$ is satisfied, where $G(A, B, C) = \inf\{G(x, y, z) : x \in A, y \in B$, and $z \in C\}$.

Main Results

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Definition

Let (X, G) be a complete G-metric space. Let A, B and C be the nonempty closed subsets of X. A mapping $T : A \cup B \cup C \to A \cup B \cup C$ is said to be cyclic G-contraction if $T(A) \subseteq B, T(B) \subseteq C$ and $T(C) \subseteq A$

$$\begin{aligned} G(Tx, Ty, Tz) \leq a_1 G(x, y, z) + a_2 G(x, Tx, Ty) \\ + a_3 G(y, Ty, Tz) \\ + (1 - (a_1 + a_2 + a_3))G(A, B, C) \end{aligned}$$

where $a_i \geq 0, i = 1, 2, 3$ and $a_1 + a_2 + a_3 < 1$, for all $x, \in A$, $y \in B$ and $z \in C$.

Main Results

Theorem

Let (X, G) be a complete G-metric space. Let A, B and C be three non-empty closed subsets of X. Let $T : A \cup B \cup C \to A \cup B \cup C$ cyclic G-contraction. Then there exists sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = G(A, B, C)$.

Main Results

Theorem

Let (X, G) be a G-metric space. Let T be G-contraction mapping. Let $x_0 \in A$ be any element and the sequence $\{x_n\}$ be defined as $Tx_n = x_{n-1}$ for all $n \geq 0$. Then $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = G(A, B, C)$. If $\{x_n\}$ has a convergent subsequence and T is continuous on A, then subsequence converges to a best proximity point.

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Time to INTERACT

Thank You