WELCOME

Group Theory

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Outline

Towers of Hanoi

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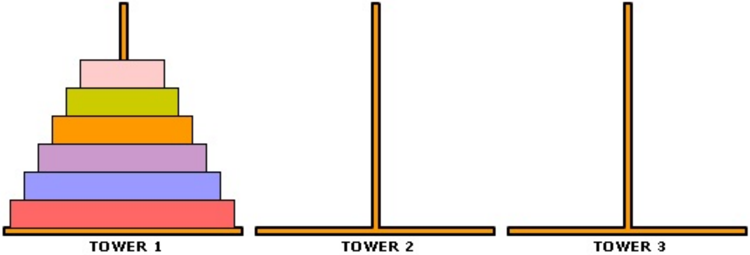
Diagram of Hanoi



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2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.
3. No disk may be placed on top of a smaller disk.

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If the legend were true, and if the priests were able to move disks at a rate of one per second, using the smallest number of moves, it would take them $2^{64} - 1$ seconds or roughly 585 billion years or 18,446,744,073,709,551,615 turns to finish, or about 127 times the current age of the sun.

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(3) The collection of districts in Tamil Nadu.

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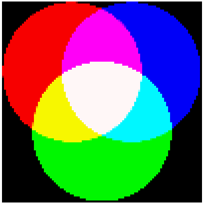

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- $4 - 4 = 0$

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Formal Statement

For all elements a, b in G, $a * b$ is in G

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For all a, b and c in G, $a * (b * c) = (a * b) * c$

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For all a in G, there exists b in G, such that
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There exists an e in the set G, such that $a * e = a$ and $e * a = a$, for all elements a in G

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- Inverse axiom : $\forall\, a \in G$ there exists an element $a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = e$.

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The order of a group G is denoted by $o(G)$.

Examples

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Solution

$(\mathbb{Z}, +)$ is an abelian. 0 is the identity element.
Inverse exists for each element in $\mathbb{Z}$.

Example 2

Show that $(\mathbb{C}, +)$ is an infinite abelian group.

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$\therefore$ $(\mathbb{C}, +)$ is an abelian group.

Example 3

Show that the set of all non-zero complex numbers is an abelian group under the usual multiplication of complex numbers.

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$\therefore\ \mathbb{C} - \{0\}$ is an abelian group.

Example 4

Show that the set of four matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ form an abelian group, under multiplication of matrices.

Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$
$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$

Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$
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Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$
$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$
Let $G = \{I, A, B, C\}$

·	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$
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Let $G = \{I, A, B, C\}$

·	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

I is the identity element. Inverse of the each element is itself.

Example 5

Show that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a a group under matrix multiplication.

Solution

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

Let $G = \{A, B\}$

Solution

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

Let $G = \{A, B\}$

(i) Closure :

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G$$

Solution

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(ii) Matrix multiplication is always associative

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity matrix

(iv) Inverse of matrix is itself.

Example 6

Show that the set G of all positive rational forms a group under the composition $*$ defined by
$a * b = \frac{ab}{3}$ for all $a, b \in G$.

Solution

(i) Since a, b in positive rational number.

$$\therefore\ a * b = \frac{ab}{3}, \quad \forall a, b \text{ in positive rational.}$$

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$\therefore\ a * b = \frac{ab}{3}, \quad \forall a, b$ in positive rational.

(ii) $a * (b * c) = a * \frac{bc}{3} = \frac{abc}{3}$

$(a * b) * c = \frac{ab}{3} * c = \frac{abc}{3}$

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(iii) $e = 3$ is the identity element.

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$(a * b) * c = \dfrac{ab}{3} * c = \dfrac{abc}{3}$

$\therefore\ a * (b * c) = (a * b) * c$

(iii) $e = 3$ is the identity element.

(iv) Let $a \in G$. Then $a^{-1} = \dfrac{9}{a}$.

Example 7

Show that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \right\},$$

where $\omega^3 = 1, \omega \neq 1$ form a group with respect to matrix multiplication.

Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, B = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},$
$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$

Solution

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, B = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},$
$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$
Let $G = \{I, A, B, C, D, E\}$

Solution Cont ···

·	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

Solution Cont · · ·

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

Solution Cont ···

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

Solution Cont · · ·

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

Solution Cont $\cdots$

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I,

Solution Cont ⋯

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I, inverse of A is B,

Solution Cont $\cdots$

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I, inverse of A is B, inverse of B is A,

Solution Cont $\cdots$

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I, inverse of A is B, inverse of B is A, inverse of C is C,

Solution Cont $\cdots$

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I, inverse of A is B, inverse of B is A, inverse of C is C, inverse of D is D

Solution Cont ⋯

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

From the table

(i) closure is verified

(ii) Matrix multiplication is always associative

(iii) I is the identity matrix

(iv) Inverse of I is I, inverse of A is B, inverse of B is A, inverse of C is C, inverse of D is D and inverse of E is E.

Example 8

Show that the set $G = \{2^n / n \in \mathbb{Z}\}$ is an abelian group under multiplication.

Solution

$G = \{2^n / n \in \mathbb{Z}\}$ to show that $(G, .)$ is an abelian group.

Solution

$G = \{2^n / n \in \mathbb{Z}\}$ to show that $(G, .)$ is an abelian group.

(i) Let $a = 2^n, b = 2^m$ then
$a \cdot b = 2^n \cdot 2^m = 2^{n+m} \in G$

Solution

$G = \{2^n / n \in \mathbb{Z}\}$ to show that $(G, .)$ is an abelian group.

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(ii) Associative property is satisfied.

Solution

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(iii) 1 is the identity element.

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$a \cdot b = 2^n \cdot 2^m = 2^{n+m} \in G$

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(v) Commutative is obvious.

Solution

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(i) Let $a = 2^n, b = 2^m$ then
$a \cdot b = 2^n \cdot 2^m = 2^{n+m} \in G$

(ii) Associative property is satisfied.

(iii) 1 is the identity element.

(iv) Inverse of 2^n is 2^{-n}

(v) Commutative is obvious.

Example 9

Prove that $< S, \cdot >$ where $S = \{1, \omega, \omega^2\}$ $1, \omega, \omega^2$ are cube roots of unity is a finite abelian group.

Solution

Let $S = \{1, \omega, \omega^2\}$

Solution

Let $S = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Solution

Let $S = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure, Associative, commutative obvious.

Solution

Let $S = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure, Associative, commutative obvious.

(ii) Identity is 1

Solution

Let $S = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure, Associative, commutative obvious.

(ii) Identity is 1

(iii) Inverse of 1 is 1, inverse of ω is ω^2 and inverse of ω^2 is ω.

Solution

Let $S = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure, Associative, commutative obvious.

(ii) Identity is 1

(iii) Inverse of 1 is 1, inverse of ω is ω^2 and inverse of ω^2 is ω.

Example 10

Prove that $< S, . >$ where $S = \{1, -1, i, -i\}$ is a set of fourth roots of unity, is a group where $i^2 = -1$.

Solution

$S = \{1, -1, i, -i\}$

Solution

$S = \{1, -1, i, -i\}$

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Solution

$S = \{1, -1, i, -i\}$

$*$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(i) The identity element is 1.

Solution

$S = \{1, -1, i, -i\}$

$*$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(i) The identity element is 1.

(ii) The inverse of 1 is 1, inverse of $1, i, -1$ and $-i$ are $1, -i, -1$ and i respectively.

Problems

Question 1

The set of all real numbers under the usual multiplication operation is not a group since

Question 1

The set of all real numbers under the usual multiplication operation is not a group since

Answer Key

(a) multiplication is not a binary operation

(b) multiplication is not associative

(c) identity element does not exist

(d) zero has no inverse

Question 1

The set of all real numbers under the usual multiplication operation is not a group since

Answer Key

(a) multiplication is not a binary operation
(b) multiplication is not associative
(c) identity element does not exist
(d) zero has no inverse

Answer is

The correct choice is (d) zero has no inverse

Question 2

If $(G, \cdot)$ is a group such that
$(ab)^{-1} = a^{-1}b^{-1}, \forall a, b \in G$, then G is a / an

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(a) commutative semi group
(b) abelian group
(c) non - abelian group
(d) None of these

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$(ab)^{-1} = a^{-1}b^{-1}, \forall a, b \in G$, then G is a / an

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(d) None of these

Answer is

The correct choice is
(b) Abelian group

Question 3

The inverse of $-i$ in the multiplicative group, $\{1, -1, i, -i\}$ is

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Answer Key

(a) 1
(b) -1
(c) i
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Answer Key

(a) 1
(b) -1
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(d) -i

Answer is

The correct choice is (c) i

Question 4

The set of integers $\mathbb{Z}$ with the binary operation $*$ defined as $a * b = a + b + 1$ for $a, b \in \mathbb{Z}$, is a group. The identity element of this group is

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Answer Key

(a) 0

(b) 1

(c) -1

(d) 12

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The set of integers $\mathbb{Z}$ with the binary operation $*$ defined as $a * b = a + b + 1$ for $a, b \in \mathbb{Z}$, is a group. The identity element of this group is

Answer Key

(a) 0

(b) 1

(c) -1

(d) 12

Answer is

The correct choice is (c) -1

Question 5

In the group $(G, \cdot)$, the value of $(a^{-1}b)^{-1}$ is

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In the group $(G, \cdot)$, the value of $(a^{-1}b)^{-1}$ is

Answer Key

(a) ab^{-1}
(b) $b^{-1}a$
(c) $a^{-1}b$
(d) ba^{-1}

Question 5

In the group $(G, \cdot)$, the value of $(a^{-1}b)^{-1}$ is

Answer Key

(a) ab^{-1}

(b) $b^{-1}a$

(c) $a^{-1}b$

(d) ba^{-1}

Answer is

The correct choice is (b) $b^{-1}a$

Question 6

If $(G, \cdot)$ is a group, such that $(ab)^2 = a^2b^2 \, \forall \, a, b \in G$, then G is a / an

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If $(G, \cdot)$ is a group, such that $(ab)^2 = a^2 b^2 \,\forall\, a, b \in G$, then G is a / an

Answer Key

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(b) abelian group
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Question 6

If $(G, \cdot)$ is a group, such that $(ab)^2 = a^2 b^2 \, \forall \, a, b \in G$, then G is a / an

Answer Key

(a) commutative semi group
(b) abelian group
(c) non - abelian group
(d) None of these

Answer is

The correct choice is (b) abelian group

Question 7

$(\mathbb{Z}, *)$ is a group with $a * b = a + b + 1 \, \forall \, a, b \in \mathbb{Z}$.
The inverse of a is

Question 7

$(\mathbb{Z}, *)$ is a group with $a * b = a + b + 1 \, \forall \, a, b \in \mathbb{Z}$. The inverse of a is

Answer Key

(a) 0
(b) -2
(c) $a - 2$
(d) $-a - 2$

Question 7

$(\mathbb{Z}, *)$ is a group with $a * b = a + b + 1 \, \forall \, a, b \in \mathbb{Z}$.
The inverse of a is

Answer Key

(a) 0
(b) -2
(c) $a - 2$
(d) $-a - 2$

Answer is

The correct choice is
(d) $-a - 2$

❁❁❁ Time to Interact ❁❁❁


Thank You