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Chapter 1

GRAPHS

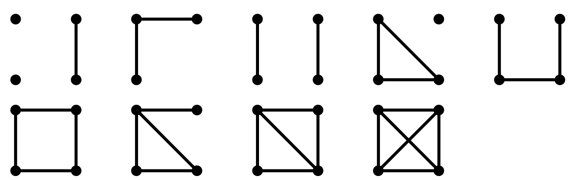
1.1 INTRODUCTION

Graph theory is a branch of mathematics which has applications in many areas like anthropology, architecture, biology, chemistry, computer science, economics, environmental conservation, psychology and telecommunication, to name a few. The list goes on and on. In a typical situation, a problem arises in a real world subject area that can be modded using graphs.Then existing theorems or algorithms are used or new ones are developed to solve the original problem. We describe the modelling process and present the basic concepts and terminology of graph theory with emphasis on the concept of distance in graphs. In addition, we describe a variety of graphs and useful operations.

1.2 GRAPHS AS MODELS

Definition 1.1. *In this section we define graphs and see how they are used. A graph* G *is a finite non-empty set* $V = V(G)$ *of* p *nodes together with an unordered pairs of distinct nodes of* V*. We say* G *has order* p *and* q*. The pair* $e = \{u, v\}$ *of nodes in* E *is called an edge of* G *and to join* u *and* v*. We write* $e = uv$ *and say* u *and* v *are adjacent and adjacent nodes are said to be neighbours. Edge* e *is incident with the two nodes* u *and* v*. A graph with* p *nodes and* q *edges is called a* $\{p,q\}$ *graph.*

And the intuition in using graphs, it is customary to represent a graph by means of a diagram and refer to it as a graph.



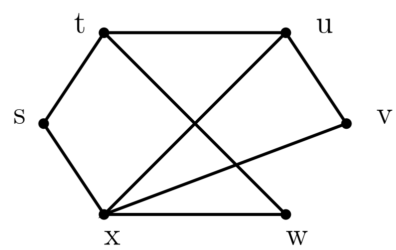
The eleven graphs with 4 graphs

Theorem 1.2. *The sum of degrees of the nodes of a graph is twice the number of edges.*

$$\sum deg v_i = 2q$$

Proof : Since each edge e is incident with two nodes, e contributes 2 to the sum of the degrees of the nodes.

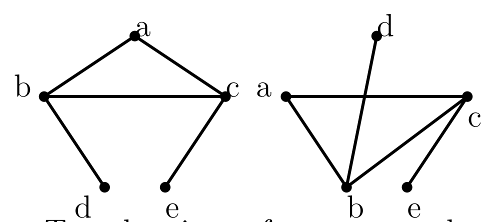
∎



A graph to illustrate adjacency and incidence

Corollary 1.3. *In any graph, the number of nodes of odd degree is even.*

The *degree sequence* is a list of the degrees of the nodes in non-increasing order. The *minimum degree* among the nodes of a graph G is denoted by $\delta(G)$ while the *maximum degree* by $\Delta(G)$ is the largest such number. Thus, the graph in above figure has degree sequence $\{4, 3, 3, 2, 2, 2\}$. so $\delta(G) = 2$ and $\Delta(G) = 4$. If all the nodes have the same degree of G and write $degG = 4$. A 3-regular graph is called *cubic*.



Two drawings of a same graph

Definition 1.4. *Two graphs G and H are said to be Isomorphic (written $G \cong H$ or sometimes $G = H$ and called equal) if there exists a one-to-one correspondence between their node sets which preserves adjacency. The three graphs G_1, G_2, G_3 are all isomorphic to each other. For example G_1 and G_2 are isomorphic under the correspondence $v_i \longleftrightarrow u_i$.*

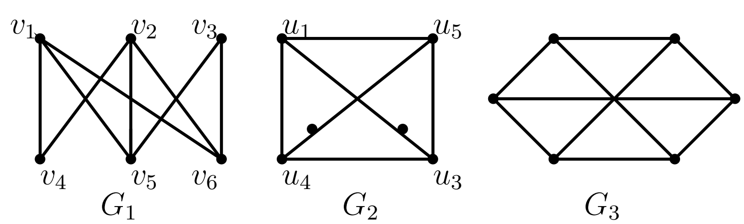
The invariant of a graph G is a number associated with G which has the same *value for any graph isomorphic to G. We now have two simple invariants and one sequences of invariants we can use to distinguish a pair of non-isomorphic graphs:*

1. the number of nodes, p

2. the number of edges, q

3. the degree sequence

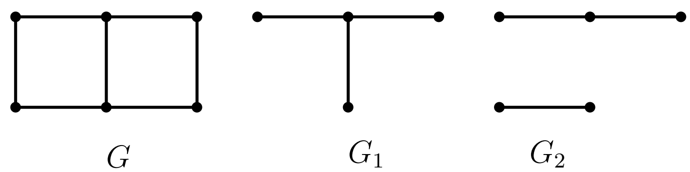
A number of variations of graphs occur in applications. A directed graph or digraph D consists of a finite non-empty set V of nodes together with a collection A of ordered pars of distinct nodes in V. The elements of A are called arcs or directed edges. A symmetric pair of arcs join two nodes u and v, one in each direction that is arcs(u, v) and (v, u). An oriented graph is a digraph with three nodes and three arcs are shown below and the last tow are oriented graphs.



Three isomorphic graphs

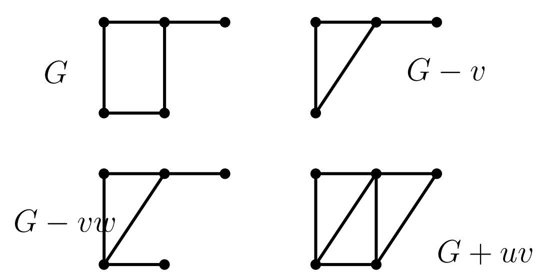
1.3 PATHS AND CONNECTEDNESS:

One of the most basic properties any graph can enjoy is that of being connected. Informally, a graph is connected if it is all in one piece. In this section, we make this concept precise and examine several fundamental classes of connected graphs: paths, trees and cycles.



A graph and its two subgraphs

Definition 1.5. *A subgraph of G is a graph having all of its nodes and edges in G. It is a Spanning subgraph if it contains all the nodes of G. If H is a subgraph of G, then G is a supergraph of H. For any set S of nodes in G, the induced subgraph$\langle S \rangle$ if and only if they are adjacent in G. In the above figure, G_1 and G_2 are subgraphs of G. Here G_1 is an induced subgraph but G_2 is not; G_2 is a spanning subgraph but not G_1.*



A graph minus a node
A graph plus or minus an edge

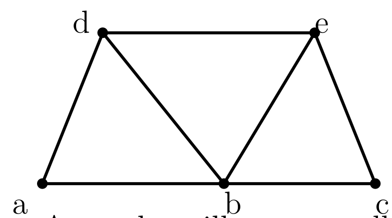
The *removal of a node* v from a graph G results in that subgraph $G - v$ consisting of all nodes of G except v and all edges are not incident with v. On the other hand, the *removal of an edge* e from G yields the spanning subgraph $G - e$ containing all the edges of G except e. Thus $G - v$ and $G - e$ are the maximal subgraphs of G not containing v and e, respectively. If u and v are not adjacent in G, the addition of the edge uv results in the smallest supergraph of G containing the edge uv and is denoted $G + uv$.

There are certain graphs for which the result of deleting an node or an edge or adding an edge in independent of the particular node or edge selected. If this is so far for a graph G, we denote the result accordingly by $G - v$, $G - e$, $G + e$. For now mention that any cycle C_n is such a graph

Definition 1.6. *A walk in a graph G is an alternating sequence of nodes and edges $v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n$ such that every $e_i = v_{i-1}v_i$ is an edge of G, $1 \leq i \leq n$. It is important to mention that the nodes need not be distinct and the same holds for the edges.*

The walk connects v_0 and v_n and is sometimes called a $v_0 - v_n$ walk.This walk has length n , the number of occurrences of edges in it. A walk is atrail if all its edges are distinct and a path if all its nodes are distinct. The walk is closed if $v_o = v_n$ and is open otherwise. A closed walk is a cycle provided its n nodes are distinct and $n \geq 3$.

Since the edges in a walk are determined uniquely by writing its successive nodes, we usually do not list the edges the edges. In labelled graph of G of the following figure a, b, e, c is a path and b, d, e, b is a cycle.



A graph to illustrate walks

Definition 1.7. *The girth of a graph G, denoted $g(G)$ is the length of the shortest cycle in G, the circumference c(G) is the length of any longest cycle. Note that these terms are undefined if G has no cycles. The distance d(u,v) between two nodes u and v in G is the minimum length of a path joining them if any; otherwise $d(u,v) = \infty$. The diameter d(G) of a connected graph G is the length of any longest geodesic. The graph G in the above figure has girth $g = 3$, circumference $c = 5$ and diameter $d = 2$.*

A graph is connected if there is a path joining each pair of nodes. A component of a graph is a maximal connected subgraph. If a graph has only one component it is connected, otherwise it is disconnected G and G_1 each have one component while G_2 has two.

Among important connected graphs of order p, a cycle is denoted by C_p and path by P_p. The complete graph K_p has every pair of its p nodes adjacent. Thus K_p has $\binom{p}{2}$ edges and is regular of degree $p - 1$.

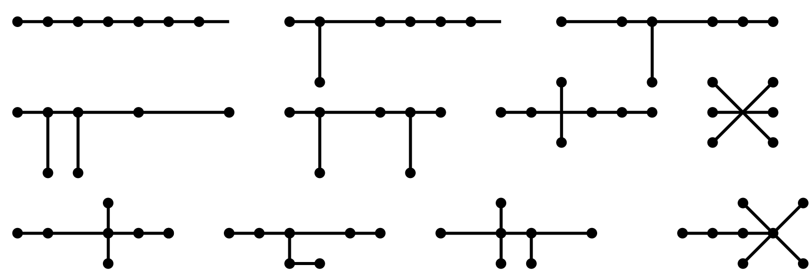
The Kelly-Ulam Conjecture:

Let G have p nodes v_i and H have p nodes u_i with $p \geq 3$ for each $i, G - v_i$ and $H - v_i$ are isomorphic, then the graphs G and H are isomorphic.

This conjecture is sometimes referred to as the reconstruction cycle because of the following view point of the problem. Draw each of the p unlabelled graphs $G - v_i$ on 3×5 card thus obtaining the deck $D(G)$ of the graph G. A *legitimate deck* is the one that can be obtained from some graph. Then the conjecture can be reformulated in terms of just one graph by asserting that any graph from which these subgraphs can be obtained by deleting one node at a time is isomorphic to

G. Thus, one may be reconstructed. The conjecture has been proven for several classes of graphs including regular graphs, disconnected graphs, and the class of graphs we discuss - trees.

Definition 1.8. *Perhaps the most important type of graph is a tree. This is so because of their applications to many different fields. Furthermore, their simplicity makes it possible to investigate a conjecture for graphs by first studying it for trees. A graph is cyclic if it has no cycles. A tree is a connected acyclic graph. Any graph without cycle is a forest, thus the components of a forest are trees. There are 11 different trees with seven nodes as shown below.*



The eleven trees with seven nodes

Theorem 1.9. *The following statements are equivalent.*

1. G is a tree.

2. Every two nodes of G are joined by a unique path.

3. G is connected and $p = q + 1$.

4. G is acyclic and $p = q + 1$.

5. G is acyclic and if any two nonadjacent nodes of G are joined by an edge E, then $G + e$ has exactly one cycle.

Proof : $(1 \Longrightarrow 2)$. Since G is connected, every two nodes are joined by a path. Let P and $P*$ be two paths joining u and v in G, and let w be the first node of P such that w is in both P and $P*$ but its successor on P is not on $P*$. If we let $w*$ be the next node on P which is also on $p*$, then the segments of P and $P*$ which are between w and $w*$ together form a cycle in G. Thus if G is acyclic, there is at most one path joining any two nodes.

$(2 \Longrightarrow 3)$. Clearly G is connected. We prove $p = q+1$ by induction. It is obvious that for graphs of one or two nodes. Assume it is true for graphs with fewer than p nodes. Suppose G has p nodes, q edges and let v be a node of degree one(there must be such a node because of the uniqueness of paths, connectedness, and $p \geq 2$) in G. Then $G - v$ has $p - 1$ nodes, one less than G, and still satisfies property 2. By the inductive hypothesis, $G - v$ has order $p - 1 = (q - 1 + 1)$. Thus the number of nodes in G is $p = q + 1$.

$(3 \Longrightarrow 4)$. Assume G has a cycle of length n. Then there are n nodes and n edges on the cycle, and for each of the $p - n$ nodes not on the cycle there is an incident edge on a geodesic to a node of the cycle. Each such edge is different, so $p \geq q$, which is a contradiction.

$(4 \Longrightarrow 5)$. Since G is acyclic, each component of G is a tree. If there are k components, then since each component has one more node than edge. $p = q+k$, so $k = 1$ and G is connected. Thus G is a tree and there is exactly one path connecting any two nodes of G. If we add an edge uv to G, that edge together with the unique path in G joining u and v forms a cycle. The cycle is unique because the path is unique.

$(5 \Longrightarrow 1)$. The graph G must be connected, for otherwise an edge e could be added joining two nodes in different components, and $G + e$ would be acyclic.

Thus G is connected and acyclic, so G is tree.

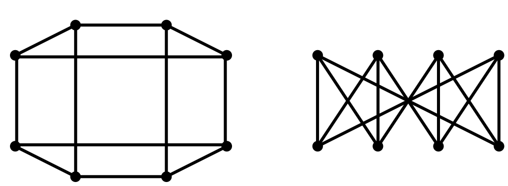
∎

Corollary 1.10. *Every non-trivial tree has at least two endnodes.*

Proof : Let P be the longest path in a non-trivial tree T and let u and v be end nodes of P. Since T is acyclic, u and v each have only one neighbour in P, and since P is a longest path they have no neighbours in $T - P$. Thus, there must be at least two nodes of degree one in non-trivial tree.

∎

Definition 1.11. *A tree is a special type of bipartite graph. A graph G is bipartite if its node set V can be partitioned into two subsets V_1 and V_2. For example the following both figures shows that the graph is bipartite. If G contains every edge joining edge joining V_1 and V_2, then G is complete bipartite graph. In this case, if V_1 and V_2 have m and n nodes, we write $G = K_{m,n}$. Obviously $K_{m,n}$ has mn edges. A star is a complete bipartite graph $K_{1,n}$. The complete n-partite graph $K(p_1, p_2, ..., p_n)$ has node set V that can be partitioned into n parts $V_1, V_2, ..., V_n$ so that V_i has p_i nodes and two nodes are adjacent if and only if they are in distinct parts. Thus, a complete bipartite graph is a complete multi-partite graph with just two parts.*



A graph and its bipartite graph

Theorem 1.12. *A graph G is bipartite if and only if all its cycles are even.*

Proof : If G is bipartite, then its node set V can be partitioned into two sets V_1 and V_2 so that every edge of G joins a node of V_1 with a node of V_2. Thus, every cycle $v_1, v_2, ..., v_n, v_1$ in G necessarily has its oddly subscripted nodes in V_1, say and the others in V_2, so that its length is even.

For the converse, we assume without loss of generality, that G is connected (for otherwise we can consider the components of G separately)Since G is connected there is an edge uv joining two nodes of V_1. Then the union of geodesics from v_1 to v and from v_1 to u together with the edge uv contains an odd cycle, a contradiction.

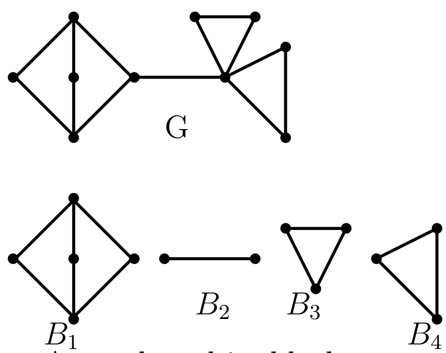
∎

1.4 CUTNODES AND BLOCKS

Some connected graphs can be disconnected by the removal of a single node called a cutnode. The distribution of such nodes is of considerable assistance in the recognition of the structure of connected graphs. Edges with the analogous cohesive property are known as bridges. The fragments of a graph held together by its cutnodes and bridges are called its blocks.

Definition 1.13. *A cutnode of a graph is a node whose removal increases the number of components, and a bridge is such a edge. Thus if v is a cutnode of a connected graph G, then $G - v$ is disconnected. A non-separable graph* *is connected, nontrivial, and has no cutnodes. A block of graph is a maximal nonseparable subgraph.*

In the following figure, v is a cutnode while w is not; edge x is a bridge but y is not and the four blocks of G are displayed. Each edge of a graph lies in exactly one of its blocks, as does each node that is not isolated or a cutnode. Furthermore, the edges of any cycle of G lies entirely within a single block.



A graph and its blocks

Theorem 1.14. *Let v be a node of a connected graph G. Then v is a cutnode of G if and only if there exist nodes u and w distinct from v such that v is on every $u - w$ path.*

Proof : If v is a cutnode in the connected graph G, then $G - v$ is disconnected. Let u and w paths in $G - v$ but there are $u - w$ paths in G since G is connected. thus every $u - w$ path in G contains v.

Conversely, if v is on every path in G joining u and w then there cannot be a path joining these nodes in $G - v$. Thus $G - r$ is disconnected, so v is a cutnode of G.

∎

Theorem 1.15. *Let e be an edge of a graph G. The following statements are equivalent.*

1. e is a bridge.

2. e is not on any cycle of G.

3. There exist nodes u and v of G such that the edge e of G is on every path joining u and v.

Theorem 1.16. *Let G be a connected graph with at least three nodes. the following statements are equivalent.*

1.

$$Gisnon-separable.2.Everytwonodesof$$

Glieonacommoncycle.

3.*Everynodeandedgeof Glieonacommoncycle.*

4.*Everytwoedgesof Glieonacommoncycle.*

5.*Giventwonodesandoneedgeof G, thereisapathjoiningthenodeswhichcontainstheedge.*

6.*Foreverythreedistinctnodesof G, thereisapathjoininganytwoofthemwhichcontainsthethird.*

7.*Foreverythreedistinctnodesof G, thereisapathjoininganytwoofthemwhichdoesnotcontainsthet*

Proof : ($1 \Longrightarrow 2$). Let u and v be distinct nodes of G, and let U be the set of nodes different from u which lie on a cycle containing u. Since G has at least three nodes and no cutnodes, it has no bridges; therefore, every node adjacent to u is in U, so U is not empty.

Suppose v is not in U. Let w be a node in U for which the distance $d(w, v)$ is minimum. Let P_0 be a shortest $w - v$ path, let P_1 and P_2 be the two $u - w$

paths of a cycle containing u and w. Since w is not a cutnode there is a $u - v$ path P' not containing w . Let w' be the node nearest u in P' in either P_1 or P_2. Without loss of generality, we assume u' is in P_1.

Let Q_1 be the $u - w'$ path consisting of the $u - u'$ subpath of P_1 and the $u' - w'$ subpath of P'. Let Q_2 be the $u - w'$ path consisting of P_2 followed by the $w - w'$ subpath of P_0. Then Q_1 and Q_2 are distinct $u - w'$ paths. Together they form a cycle, so w' is in U. Since w' is on a shortest $w - v$ path, $d(w', v) < d(w, v)$. This contradicts our choice of w, proving that u and v do lie on a cycle.

$(2 \Longrightarrow 3)$. Let u be a node and vw be an edge of G. Let Z be a cycle containing u and v. A cycle Z' containing u and vw can be formed as follows. If w is on Z, then Z' consists of vw together with the $v - w$ path P not containing v, since otherwise v would be a cutnode by previous before theorem. Let u' be the first node of P in Z. Then Z' consists of vw followed by the $w - u'$ subpath of P and the $u' - v$ path in Z containing u.

$(3 \Longrightarrow 4)$. The proof is analogous to the proceeding one, and the details are omitted.

$(4 \Longrightarrow 5)$ Any two nodes of G are incident with one edge each, which lie on a cycle by (4). Hence ant two nodes of G lie on a cycle, and we have (2), so also (3). Let u and v be distinct nodes and e an edge of G. By (3), there are cycles Z_1 containing u and e and Z_2 containing v and e. Thus, we need only consider the case where v is not on Z_1 and u is not on Z_2. Begin with u and proceed along Z_1 until reaching the first node w of Z_2, then take the path on Z_2 joining w and v which contains e. This walk constitutes a path joining u and v that contains e.

$(5 \Longrightarrow 6)$ Let u, v and w be distinct nodes of G, and let e be any edge incident with w. By (5), there is a path joining u and v which contains e, and hence must contain w.

($6 \Longrightarrow 7$) Let u, v and w be distinct nodes of G. By statement (6), there is a $u - w$ path P containing v. The $u - v$ subpath of P does not contain w.

($7 \Longrightarrow 1$) By statement (7), for any two nodes u and v, no node lies on every $u - v$ path. Hence, G must be nonseparable.

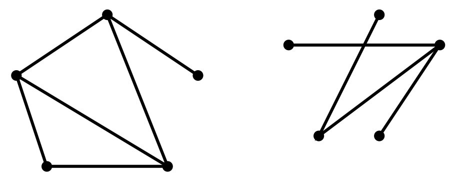
∎

Theorem 1.17. *If G is nonseparable with $\delta(G) \geq 3$, then there is a node v such that $G - v$ is also nonseparable.*

1.5 GRAPH CLASSES AND GRAPH OPERATIONS

When a new concept is developed in a graph theory, it is often first applied to particular classes of graphs. Afterwards, more general graphs are studies and theorems follow. In the previous sections, we encountered the paths, cycles, trees ans bipartite graphs. Many interesting graphs are obtained by combining pairs of graphs or operating on a single graph in some way.

Definition 1.18. *The complement$\bar{G}$ of a graph G has $V(G)$ as its node set, but two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A graph and its complement are shown below. The graphs $\bar{}K_p$ are called totally disconnected and are regular of degree 0.*



A graph and its complement

Theorem 1.19. *For any graph G of order 6, G or $\bar{G}$ contains a triangle.*

Proof : Let v be a node of a graph G with six nodes. Since v is adjacent either in G or $\bar{G}$ to at least half of the five other nodes of G, we can assume without loss of generality that there are three nodes u_1, u_2, u_3 adjacent to v in G. If any two of these nodes are adjacent, then they are two nodes of a triangle whose third node is v. If no two of them are adjacent in G, then they are the nodes of a triangle in $\bar{G}$.

∎

Theorem 1.20. *If G is disconnected, then $\bar{G}$ is connected.*

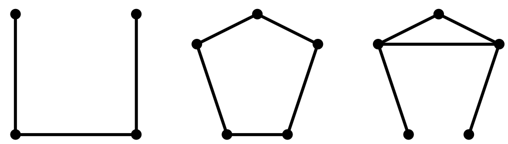
Definition 1.21. *Self-complementary graph is isomorphic with its complement. Our first result about self-complementary graphs specifies their first order.*

Theorem 1.22. *If G is self-complementary, then $p = 4n$ or $4n + 1$.*

Theorem 1.23. *If $d(G) \geq 3$, then $d(\bar{G}) \leq 3$.*

Proof : Let x and y be any two nodes in $\bar{G}$. Since $d(G) \geq 3$, there exist nodes u and v at distance 3 in G. Hence uv is an edge in $\bar{G}$. Since u and v have no common neighbour in G, both x and y are adjacent to u or v in $\bar{G}$. It follows that $d(x, y) \leq 3$ in $\bar{G}$, and hence $d(\bar{G}) \leq 3$.

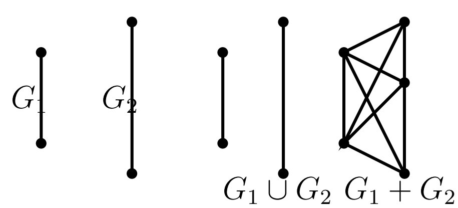
∎



The smallest nontrivial self-complementary graphs

For any connected graph G, we write nG for the graph with n-components each isomorphic with G. Then every graph can be written in the form $\cup n_i G_i$ with G_i different from G_j for $i \neq j$. There are several operations on G_1 and G_2 whose set of nodes is the Cartesian product $V_1 \times V_2$. These include the product and the composition.

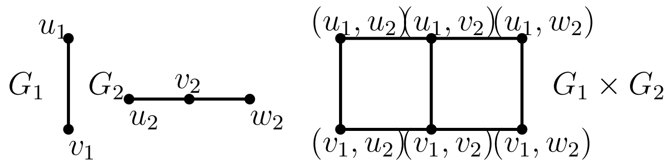
To define the *cartesian product* $G_1 \times G_2$ consider any two nodes in $G_1 \times G_2$ whenever $[u_1 = v_1 and u_2 v_2 \in E(G_2)]$ or $[u_2 = v_2 and u_1 v_1 \in E(G_1)]$.



The union and the join of two graphs

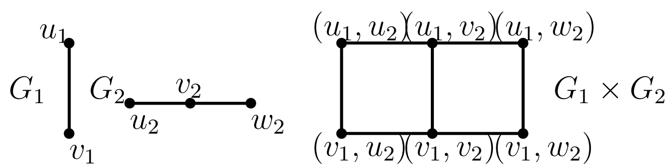
Corollary 1.24. *Every non trivial self-complementary graph has diameter 2 or 3.*

Definition 1.25. *In this section, we define the graphs G_1 and G_2 have disjoint node sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Their union $G = G_1 \cup G_2$ has expected, $V = V_1 + V_2$ and $E = E_1 + E_2$. Their join is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 and V_2. These operations are illustrated in the following figure with $G_1 = K_2 = P_2$ and $G_2 = K_{1,2} = P_3$.*



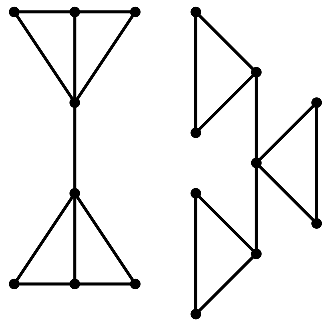
The produt of two graphs

Definition 1.26. *The corona $G_1 \circ G_2$ of graph G is the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^th node of G_1 to every node in the i^th copy of G_2. For the graphs $G_1 = K_2$ and $G_2 = P_3$, the two different coronas $G_1 \circ G_2$ and $G_2 \circ G_1$ are shown below.*



The product of two graphs

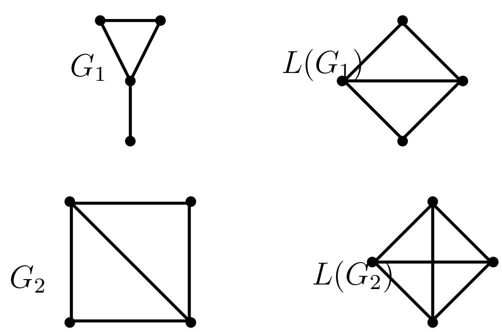
An important class of graphs now known as hypercubes are most naturally expressed in terms of products. The *n-cube* Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. Thus Q_n has 2^n nodes which may be labelled as $a_1a_2...a_n$, where each a_i is either 0 or 1. Two nodes of Q_n are adjacent if their binary sequences differ in exactly one place.



The two different coronas of two graphs

Definition 1.27. *The square G^2 of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever $d(u, v) \leq 2$ in G. The higher powers $G^3, G^4, ...$ of G are defined similarly. Powers of graphs have been studied mostly in connection with hamiltonicity and chordal graphs.*

Definition 1.28. *A clique of a graph is a maximal complete subgraph. The clique graph $K(G)$ of a given graph G has the cliques of G as its nodes and two nodes of $K(G)$ are adjacent if the corresponding cliques intersect. Not every graph is the clique graph of some graph.*



Graphs and their line graphs

Definition 1.29. *Let graph G have at least one edge. The set of nodes of line graph of G, denoted $L(G)$ consists of the edges of G with two nodes of $L(G)$*

adjacent whenever the corresponding edges of G are. Two examples of graphs and their line graphs are given in the following figure. Note that in this figure $G_2 = L(G_1)$, so that $L(G_2) = L(L(G_1))$. We write $L^2(G) = L(L(G))$, and in general the iterated line graph $L^n(G) = L(L^{n-1}(G))$.

A graph G is a line graph if it is isomorphic to the line graph $L(H)$ of some graph H. For example $K_4 - e$ is a line graph. On the other hand, we now verify that $K_{1,3}$ is not a line graph. Assume $K_{1,3} = L(H)$. Then H has four edges a, b, c, d since $K_{1,3}$ has four nodes. In H one of the edges, say a, is adjacent with the other three edges, while none of b, c, d are adjacent. Since a has only two endnodes, at least one pair of b, c, d must be adjacent to a at a single node, making that pair of edges adjacent to one another as well, a contradiction. So $K_{1,3}$ is not a line graph. By the same reasoning $K_{1,3}$ cannot be an induced subgraph or a line graph.

Theorem 1.30. *A graph G is a line graph G is a line graph if and only if the edges of G can be partitioned into complete subgraphs in such a way that no node lies in more than two of the subgraphs.*

Proof : Let G be a line graph of H. Without loss of generality, we assume that H has no isolated nodes. Then the edges in the star at each node of H induce a complete subgraph of G and every edge lies in exactly one such subgraph. Since each edge of H belong to the stars of exactly two nodes of H, no node of G is in more than two of the complete subgraphs.

Given a partition of the edges of a graph G into complete subgraphs $S_1, S_2, ..., S_n$ such that no node lies in more than two of the subgraphs, we construct a graph

H whose line graph is G. The nodes of H correspond to the set S of subgraphs $S_1, S_2, ..., S_n$ together with the set U of nodes belonging to only one of the subgraphs S_i. Thus $S \cup U$ is the node set of H and two of these nodes are adjacent whenever they have nonempty intersection.

∎

Theorem 1.31. *G is a line graph if and only if*

1. $K_{1,3}$ is not an induced subgraph of G, and

2. if $K_4 - e$ is an induced subgraph of G, then at least one of the two triangles in $K_4 - e$ is even.

Corollary 1.32. *Graph G is a line graph if and only if none of its nine graphs of following figure is an induced subgraph of G.*

Proof : Using the above theorem, we see that $K_{1,3}$ is not an induced subgraph of a line graph G. Suppose $K_4 - e$ is an induced subgraph of G. Then to find other forbidden subgraphs,check possible adjacencies among odd, contradicting to the above theorem . For example, if some node v is adjacent to both of the degree of nodes of degree two in $K_4 - e$ and no others, then both triangles are odd so G is not a line graph. In this case, we get the second graph of the above figure. If a node v is adjacent to all the nodes of $K_4 - e$ again both triangles are odd so G is not a line graph and we find the third forbidden subgraph. If nodes u and v are each adjacent to one of the nodes of degree 2 and no other nodes in $K_4 - e$, both the triangles are odd so G is not a line graph and we find the fourth forbidden graph. Each of the other forbidden subgraphs is found in a similar manner using the above theorem.

∎

Chapter 2

CENTERS

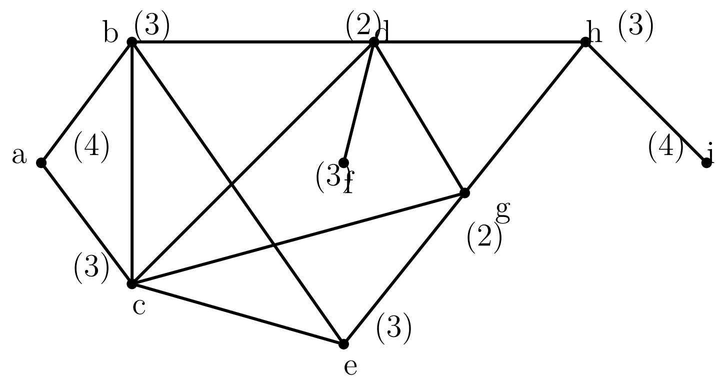
2.1 Introduction

Facility location problems deal with the task of choosing a site subject to some criteria. For example, in determining where to locate an emergency facility such as hospital or fire station, we would like to minimize the response time between the facility and the location of a possible emergency. In deciding the position for a service facility such as a post office, power station or employment office, we want to minimize the total travel time for the people. Each of these situations deals with the concept of centrality. However, the type of center differs for each of the examples mentioned. Centrality questions are now examined using graphs and distance concepts.

2.2 THE CENTER AND ECCENTRICITY

Let G be a connected graph and let v be a node of G. The *eccentricity* $e(v)$ of v is the distance to a node farthest from v. Thus

$$e(v) = max\{d(u, v) : u \in V\}$$



A graph and its ecentricities

The *radius* $r(G)$ is the minimum eccentricity of the nodes, whereas the *diameter* $d(G)$ is the maximum eccentricity. Now v is a *central node* if $e(v) = r(G)$ and the *center* $C(G)$ is the set of all central nodes. Thus, the center consists of all nodes having maximum eccentricity. Node v is a *peripheral node* if $e(v) = d(G)$ and the *periphery* is the set of all such nodes. For node v, each node at distance $e(v)$ from v is an *eccentric node* for v. These concepts are illustrated in above figure. where the eccentricity of each node is shown in parenthesis. Graph G has radius 2, diameter 4 and central nodes d and g; nodes f and i are eccentric nodes for e.

A basic result concerning centers is the classical theorem of Jordan. When $p(T) \geq 3$, let T' be the subtree of T obtained by removing all end nodes of

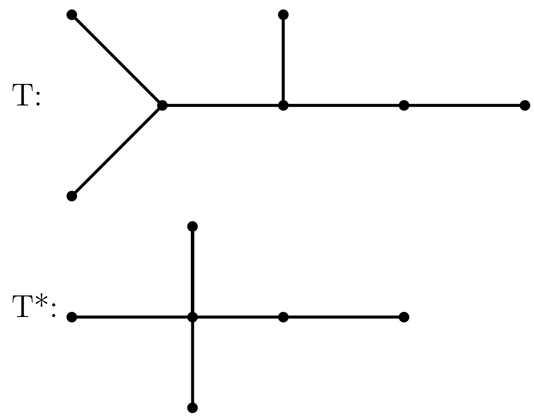
T. A *caterpillar* is atree for which the nodes are not endnodes induce a path.

Theorem 2.1. *The center of a tree consists of either a single node or a pair of adjacent nodes.*

Proof : The result is trivial for the trees K_1 and K_2. We show that any other tree T has the same center as the tree T'. Clearly, for each node v of T, only an end node can be an eccentric node for v. Thus, the eccentricity of each node in T' will be exactly one less than the eccentricity of the same node in T. Hence the nodes with minimum eccentricity in T' are the same nodes of minimum eccentricity in T, that is T and T' have the same center. If the process of removing end nodes is repeated, we obtain successive trees having the same center as T. Since T is finite, we eventually obtain a subtreee of T which is either K_1 or K_2. In either, case, the nodes in this ultimate tree constitute the center of T which thus consists of a single node or a pair of adjacent nodes.

∎

A tree with one central node is called a*central tree* and one with two central nodes is called *bicentral*



A central tree T and bicentral tree T*

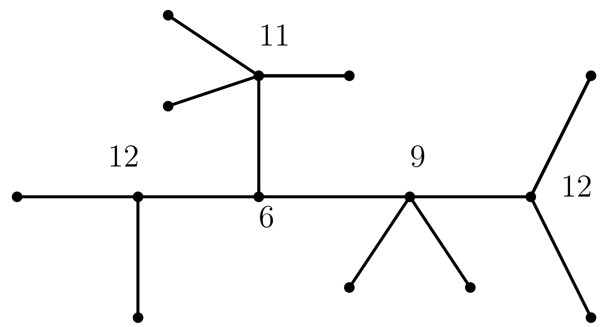
Theorem 2.2. *The center $C(G)$ of any connected graph G lies within a block of G.*

Proof : Suppose the center $C(G)$ of a connected graph G lies in more than one block. Then G contains a cutnode v such that $G - v$ has components G_1 and G_2 each of which conatins a central node of G. Let u be a eccentric node of v and let P be an $u - v$ path of length $e(v)$. Then P contains no node from at least one of G_1 and G_2, say G_1. Let w be acentral node of G_1 and let P' be a $w - v$ geodesic in G. Then $e(w) \geq d(w, v) + d(v, u) \geq 1 + e(v)$. So w is not a central node, a contradiction. Thus all central nodes must lie in a single block. ∎

Definition 2.3. ***The Centroid:***

A branch at a node v of a tree T is a maximal subtree containing v as an endnode. Thus, the number of branches at v is $degv$. The weight at a node v of T is the maximum number of edges in nay branch at v. The weights at the non endnodes of the tree in following figure are indicated. Of course, the weight at each endnode is 13, the number of edges.

A node v is a centroid node of a tree T if v has minimum weight, and the centroid *of T consists of all such nodes. Centroids have not been widely studies because, until recently, the were only defined for trees.*



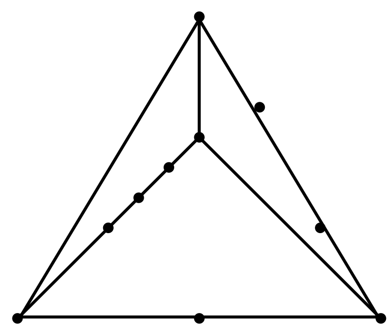
The weights at the nodes of a tree

Theorem 2.4. *Every tree has a centroid consisting of either one node or two adjacent nodes.*

Slater extended the concept of a centroid so that it is defined for all connected graphs. For a given pair of nodes u and v , let $c(u)$ be the number of nodes which are closer to u than to v, and let $c(v)$ be the number of nodes which are closer to v than to u. Let $f(u, v) = c(u) - c(v)$ and le $g(u) = \Sigma_{v \in V-u} f(u, v)$. The *centroid of a graph G* is the set of all nodes for which $g(u)$ is maximum.

Definition 2.5. ***Structural Results:***

A graph is planar if it can be drawn in the plane with no crossing edges. Two graphs are homeomorphic if they can both be obtained from the same graph by a sequence of subdivisions of edges. For example, any two cycles are homeomorphic, and a graph homeomorphic to K_4 is displayed.



A homeomorph of K_4

Definition 2.6. *The central subgraph $\langle C(G)\rangle$ of a graph G is the subgraph induced by the center. A structural result for centers is of the following form: if G is a certain type of graph then the central subgraph of G must have a particular structure. Jordan result asserts that if T is a tree, then $\langle C(T)\rangle$ is isomorphic to K_1 or K_2.*

A graph is outerplanar, if it can be drawn in the plane with all nodes in the exterior boundary; it is maximal outerplanar if no edge can be added without destroying its outerplanar property.

Theorem 2.7. *If G is a maximal outerplanar graph, then its central subgraph $\langle C(G)\rangle$ is isomorphic to one of the seven graphs.*

Definition 2.8. *A graph is chordal if every cycle of length greater than 3 has a chord. Every tree is a chordal graph and a maximal outerplanar graph is also chordal. The structure of the central subgraph has also been considered for chordal graphs.*

When an edge is added to a graph, the eccentricities of the nodes may be affected. A graph G is diameter-maximal if for A graph G is a unique eccentric node graph if each node in G has exactly one eccentric node.

Theorem 2.9. *A connected graph G is diameter-maximal if and only if*

1. G has a unique pair of eccentric peripheral nodes u and v.

2. the set of nodes at each distance k from u induces a complete graph

3. every node at distance k is adjacent to every node at distance $k+1$.

A disconnected graph is maximal if and only if $G = K_m \cup K_n$

Corollary 2.10. *Every diameter-maximal graph with odd diameter is a unique eccentric node graph.*

Proof : Let G be a diameter-maximal graph with odd diameter d. By the above theorem, G has two peripheral nodes u and v . for each node w in G with $d(u, w) \geq (d-1)/2$, u is its eccentric node; v is the unique eccentric node for all other nodes in G. Thus G is a unique eccentric node graph. ∎

Theorem 2.11. *If G is connected chordal graph, then*

$$d/2 \leq r \leq d/2 + 1$$

Corollary 2.12. *If G is connected, self-centered, chordal graph with radius r, then $r = 1$ or 2.*

Proof : Since G is self-centered, $r(G) = d(G)$. When r is even, the above theorem gives $r \leq r/2 + 1$,so $r = 2$. In the same way, we get $r = 1$ when r is odd. ∎

Theorem 2.13. *A unique eccentric node graph G is self centered if and only if each node of G is eccentric.*

Proof : Let G be a self centered unique centered node graph. For an arbitrary node v, denote its eccentric node by $v*$, so $d(v, v*) = r(G)$. Then v is the eccentric node for $v*$. Thus, each node of G is an eccentric node.

For the converse, we are given that each node v of an unique ecccentric node graph G is an eccentric node. We first show that $(v*)* = v$. Suppose not, and without loss of generality, assume u has least eccentricity among eccentric nodes v. Then $u = x$ for some node x. Note that $e(x) \leq e(u)$. If $e(x) = e(u)$ then $u* = x$ so $(u*)* = x* = u$, a contradiction. Thus assume $e(x) < e(u)$. Then $(x*)* = u* \neq x$ and $e(x) < e(u)$ contrary to the choice of u. Thus $(v*)* = v$ for each v in G and $e(v) = e(v*)$.

Suppose $r(G) < d(G)$. Then some pair of adjacent nodes w and v satisfy $e(w) < e(v)$. Their some pair of adjacent nodes w and v satisfy $e(w) < e(v)$. Their eccentric nodes satisfy $e(w*) = e(w) < e(v) = e(v*)$, so $w*$ and $v*$ are distinct. Since $w*$ is unique for w, $d(w, v*) < d(w, w*)$ which gives

$$d(v, v*) \leq d(v, w) + d(w, v*) = 1 + d(w, v*) < 1 + d(w, w*)$$

So $e(v) = d(v, v*) < 1 + d(w, w*) = 1 + e(w)$. Since $e(w)$ and $e(v)$ are integers, $e(v) \leq e(w)$ a contradiction. Hence we must have $r(G) = d(G)$ that is, G is self-centered. ∎

Theorem 2.14. *For a tree T, $f(T) = 0$ only for $T = K_1$ or K_2; $f(T) \neq 1 or 3$. If $p \geq 3$, then $f(T) = 2$ if and only if all end nodes of T have the same eccentricity.*

2.3 THE MEDIAN

The center of a graph is important in applications involving emergency facilities where time to each single location in the region is critical. Sippose, instead we consideer a service facility such as a post office, bank or power station. When deciding where to locate a post office, we want to minimise the average distance that a person serviced by the post office must travel. This is equivalent to minimizing the total distance travelled by all people within the district.

Let G be a connected graph. The *status s(v)* of a node v in G is the sum of the distances from v to each other node in G. The *median M(G)* of a graph G is the set of nodes with minimum status. The *m inimum status ms(G)* of a graph G is the value of minimum status; the *total status ts(G)* is sum of all the status values. These concepts are illustrated in a following figure. The number near each node is its status. The minimum status of G is 8, the total status is 70, and the median consists of nodes b, d and e.

Theorem 2.15. *Node v is a centroid node of a tree T if and only if it is a median node.*

.

Theorem 2.16. *The median $M(G)$ of any connected graph G lies within the block of G.*

Theorem 2.17. *For each node v of a connected (p, q)- graph G,*

$$p - 1 \leq s(v) \leq (p-1)(p+2)/2 - q$$

and these bounds can be achieved for each q,$p-1 \leq q \leq \binom{p}{2}$

Proof : The easy lower bound is achieved by any (p, q)- graph with some node having degree $p - 1$. We use induction q to prove the upper bound holds. Since G is connected, begin with $q = p - 1$, so G is a tree. For any node v, let d_i be the number of nodes at distance i from v. Then,

$$s(v) = \Sigma i d_i \text{ and } \sigma d_i = p - 1$$

Note that if $d_i = 0$, then $d_{i+1} = 0$. Thus the sum $\sigma i d_i$ is maximum when $d_i = 1$ for each i, so that
$s(v) = \sum_{i=1}^{p-1} = \frac{p(p-1)}{2} = \frac{(p-1)(p+2)}{2} - (p-1)$
By the inductive hypothesis, the upper bound hold for any connected (p, q)-graph. Let v be an node in a connected $(p, q+1)$-graph G. Then G is not a tree, so it contains a cycle. Let u be a node in a cycle C such that $d(u, v)$ is minimum for such nodes . Let w be a node on C adjacent to u and consider $G - uv$. This graph is connected (p, q)-graph. Because of the choice of u as a closest node to v in a cycle, $d(v, w)$ is greater in $G - uw$ than in G. By inductive hypothesis,

$$s(v) \leq -1 + \frac{(p-1)(p+2)}{2} - q = \frac{(p-1)(p+2)}{2} - (q+1)$$

so the upper bound holds.
To show that the upper bound can be achieved for each value of q, $p-1 \leq q \leq \binom{p}{2}$, let t be the largest integer for which $n = q - p + 1 - t(t-3)/2$ is non-negative.

Let G be the sequential join

$$K_1 + K_1 + ... + K_1 + K_n + K_{t-n}$$

where there are $(p-t)K_1's$. This graph has p nodes and

$$(p-t-1) + n + \binom{n}{2} + n(t-n) + \binom{t-n}{2}$$

edges. After simplifying and substituting for n, we find G is a (p,q)-graph. The node v of degree 1 in G has status

$$s(v) = \sum_{i=1}^{p-t-1} i + (p-t)n + (p-t+1)(t-n) = \frac{(p-1)(p+2)}{2} - q$$

∎

Definition 2.18. *Analogous to self-centered graphs are the self median graphs in which all nodes have the same status.*
The median subgraph of a graph G is the induced subgraph $\langle M(G)\rangle$. If G is disconnected, then all the nodes in G have the same status, so G is self median and no embedding is required.Thus we may restrict attention to connected graphs.

Theorem 2.19. *Every graph G has a supergraph H whose median subgraph is isomorphic to G.*

Proof : Let $V(G) = v_1v_2...v_p$. Then form H as follows: add p new nodes $v_1'v_2'...v_p'$ then join v_i' to v_i and to all nodes of G not adjacent to v_i. It is easy to check that in H, $s(v_i) = 3p-2$ while $s(v_i') = 3p-2+d_i$ where d_i is the degree of v_i in G. Since $d_i \geq 1$,$\langle M(H)\rangle = G$

∎

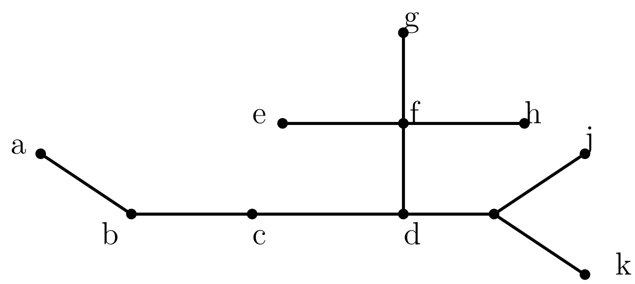
2.4 CENTRAL PATHS

If a superhighway is to be built connecting two major metropolitan areas so that it has a total of ten exists serving to towns in between, along what path should the highway be built and where should the exists be located to be most convenient to the largest number of people? If the towns all have the same political clout, the highway will be designed to minimise the maximum distance from various towns to the closest exit.

There are other situations such as the installation of natural gas pipelines or pipelines of irrigation, where one may want to find the path of all nodes in a graph are close too. In this section, we discuss several concepts which involve minimising the distance to a path in a graph.

Definition 2.20. *Let G be a graph and let W be a subgraph of G. For any node v in G, the distance $d(v,W)$ from v to W is the minimum distance from v to a node in W. The eccentricity of W, $e(W)$ is the distance to a node farthest from W.Thus $e(W) = max d(v,W)$ for v in G.*

The weight$b(W)$ of a set W of nodes in a graph G is the number of nodes in the largest component of $G - W$. The path centroid of a graph G isa path with minimum weight having minimum length among such paths.



A tree to illustrate central paths

Theorem 2.21. *The path center of a tree T is unique and it contains the center $C(T)$.*

Theorem 2.22. *The path centroid of a tree T is unique and it contains centroid of T*

Proof : Let P be apath centroid and let k be the weight of the centroid node in tree T. But we know that every tree has a centroid consisting of either one or two adjacent nodes, the centroid consists of one or pair of adjacent nodes. Assume P does not contain the centroid. Then P is a subgraph of one of the branches at a centroid node and $b(P) \geq k + 1$. This is a contradiction since the subgraph induced by the centroid is a path with smaller branch weight. Thus, any path centroid contains the centroid.
Let W be the centroid of T, and let k be the weight of $w \in W$. Suppose $\langle V(T) - W \rangle$ contains three or more components with k nodes. Since a path P through W contains nodes from at most two of these components, $b(P) \geq k$. In this case, the shortest path is $\langle W \rangle$. If $\langle V(T) - W \rangle$ contains just two components with k nodes,let u and v be the nodes from those components which are adjacent to a node in W. With u and v are path centroid P, $b(P) < k$, whereas if either u or v is not in P, then $b(P) \geq k$. Thus both u and v are in the path centroid.

Note that u and v each have minimum weight among the nodes adjacent to the centroid.

Suppose a path $P = v_1 v_2 ... v_n$ is known to be in the path centroid of $T (n \geq 2)$. Let X_i be the set of nodes in $V(T) - V(P)$ which are adjacent to v_i If $u \in X_1$ and $b(u) < b(v)$ for all $v \in \bigcup X_i - u$, then $P' = uv_1 v_2 ... v_n$ has $b(P') < b(P)$,so u is in the path centroid. If $u_1 \in X_1$ and $u_n \in X_n$ such that $b(u_1) = b(u_n)$ for all $v \in \bigcup X_i - \{u_1, u_n\}$, then $P'' = u_1 v_1 v_2 ... v_n u_n$ and both u_1 and u_n are in path centroid. In all other cases, P cannot be extended to a path with similar weight.

∎

Definition 2.23. *The status $s(P)$ of a path P in a graph G is the sum of the distances $d(c, V(P))$ for all $v \in V(G)$.A path with minimum status is a core or path median of G.*

For each v in G , let $t(v)$ be the maximum difference between $s(v)$ and $s(P)$ where P is a nontrivial path woith v as endnode. A node v in G for which $t(v)$ is minimum is called a pit node and the set of such nodes is the pit in G.

2.5 OTHER GENERALISED CENTERS

The central paths of previous section are one type of generalised center. They are special cases of more general classes of problems- n-centers, n-medians, n-centroids. When locating the set S of 3 fire houses to protect retirement communities, this is an n-center problem. A path center is an n-center for which the n nodes in S form a path. The problem where to locate a pizza store is an n-median problem. here we discuss n-centers, n-medians, the cutting center, tha path centrix, and several other generalized centers.

Definition 2.24. *The cutting number $c(v)$ of a node v in a connected graph G is the number of pairs of nodes $\{u, w\}$ such that u and w are in different components of $G - v$. The cutting center $CC(G)$ of a graph G is the set of all nodes with maximum $c(v)$; a node in $CC(G)$ is called cutting center node. Clearly, $c(v) > 0$ if and only if v is a cutnode. Cutting centers have been studied mainly for trees, where every non node has positive cutting number.It has also been shown that there are trees with an arbitraily large cutting center as well as trees with two cutting center nodes which are arbitrarily far apart from one another.*

Chapter 3

CONNECTIVITY

3.1 INTRODUCTION

Computer and telecommunication networks are often modeled by graphs. It is useful to know the reliability of a telecommunication network. That is, if one or two pieces of equipment fail, is it still possible for communication to proceed uninterupted? Network reliability problems are modelled by graphical networks where a number associated with each node and each edge represents the probability that the piece of hardware or connecting lines without fail. A related concept vulnerability which is the susceptibility of a network to successful attack by adversaries

The connectivity of a graph in a particularly intutive area of graph theory and extends the concepts of cutnode, bridge and block. Two variants called connectivity and edge-connectivity are used in deciding which two graphs are more connected.

3.2 CONNECTIVITY AND EDGE-CONNECTIVITY

Definition 3.1. *The connectivity* $\kappa = \kappa(G)$ *of a graph* G *is the minimum number of nodes whose removal results in a disconnected or trivial graph.Thus the connectivity of a disconnected graph is 0, while the connectivity of a connected graph with a cutnode is 1. The complete graph* K_p *cannot be disconnected by removing* $p-1$ *nodes; therefore,* $\kappa(K_p) = p-1$
Analogously, the edge-connectivity $\kappa' = \kappa'(G)$ *of a graph* G *is the minimum number of edges whose removal results in disconnected or trivial graph. Thus* $\kappa'(K_1) = 0$ *and the edge-connectivity of a disconnected graph is 0, while that of a connected graph with a bridge is 1.*

Theorem 3.2. *For any graph* G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Proof : We first verify the second inequality. If G has no edges, then $\kappa' = 0$. Otherwise, a disconnected graph results when all the edges incident with a node of minimum degree are removed. In either case $\kappa' \leq \delta$.
To obtain the first inequality, various cases are considered. If G is disconnected or trivial, then $\kappa = \kappa' = 0$. If G is connected and has a bridge e, then $\kappa' = 1$.In this case $\kappa = 1$ since either G has a cutnode incident with e or G is K_2. Finally,suppose G has $\kappa' - 1$ of these edges produces a bridge $e = uv$. For each of these $\kappa' - 1$ edges, select an incident node different from u or v. If the removal of these nodes produces a disconnected grpah then $\kappa < \kappa'$; if not , then $e = uv$ is a bridge, and hence the removal of u or v will result in either a disconnected or trivial graph, so $\kappa \leq \kappa'$ in every case.

∎

Theorem 3.3. *For all integers a, b, c such that $0 < a \leq b \leq c$, there exists a graph G with $\kappa(G) = a, \kappa'(G) = b$, and $\delta(G) - c$*

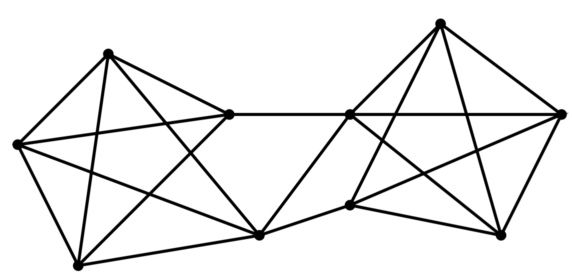
Theorem 3.4. *If a graph G has p nodes and the minimum degree $\delta(G) \geq \lfloor p/2 \rfloor$, then $\kappa'(G) = \delta(G)$*

For example, if G is regular degree $r \geq \lfloor p/2 \rfloor$, then $\kappa'(G) = r$. In particular, $\kappa'(K_p) = p - 1$.

Theorem 3.5. *Among all graphs with p nodes and q edges, the maximum connectivity is 0 when $q < p - 1$, and is $\lfloor 2q/p \rfloor$ when $q \geq p - 1$*

Proof : Since the sum of the degrees of the nodes in any (p, q)- graph is $2q$, the average degree is $2q/p$. Therefore, $\delta(G) \leq \lfloor 2q/p \rfloor$, so $\kappa(G) \leq \lfloor 2q/p \rfloor$ by theorem 3.1. To show that this value can be actually be attained, an appropriate family of graphs can be constructed.

∎



A graph with $0 \leq \kappa \leq \kappa' \leq \delta$

Corollary 3.6. *The maximum edge-connectivity of a (p, q)-graph equals the maximum connectivity.*

Definition 3.7. *A graph G is n-connected if $\kappa(G) \geq n$ and n-edge connected if $\kappa'(G) \geq n$. Thus a non trivial graph is 1-connected if and only if it is connected and 2-connected if and only if it is non separable graph that is not 2-connected.*

Theorem 3.8. *If G is connected, $n \geq 2$, then every set of n nodes of G lie on a cycle.*

Theorem 3.9. *A graph G is 3-connected if and only if it is a wheel or can be obtained from a wheel by a sequence of operations of the following two types:*
1. the addition of a new edge.
2. the replacement of a node v having at least 4 by a pair of adjacent nodes v_1, v_2 such that in the resulting graph, each node is joined to exactly one of v_1 and v_2 and deg $v_1 \geq 3$ and deg $v_2 \geq 3$.

Theorem 3.10. *If G is connected, $n \geq 2$, then its line graph $L(G)$ is also n-connected.*

Proof : Let G be n-connected and suppose that $\kappa(L(G)) < n$. Then removing $\kappa < n$ nodes from $L(G)$ will produce either a disconnected or trivial graph. Each node of $L(G)$ corresponds with an edge of G with two nodes of $L(G)$ adjacent if and only if the corresponding edges of G are incident. Thus, by removing

$\kappa < n$ edges from G, we can produce either a disconnected or trivial graph, that is, $\kappa'(G) < n$. But from thm 3.1, implies $\kappa < \kappa' < n$, a contradiction. So $\kappa(L(G)) \geq n$, that is, $L(G)$ is n-connected.

∎

Theorem 3.11. *If G is n-connected, $n \geq 2$, then every set of n nodes of G lie on a cycle.*

By taking G to be a cycle C_n, it is seen that the converse is not true for $n > 2$.

Theorem 3.12. *A graph G is 3-connected if and only if it is a wheel or can be obtained from a wheel by a sequence of operations of the following two types:*
1. The addition of new edge.
2. The replacement of a node v having degree at least 4 by a pair of adjacent nodes v_1, v_2 such that in the resulting graph, each node is joined to exactly one of v_1 and v_2 and $deg v_1 \geq 3$ and $deg v_2 \geq 3$.

Theorem 3.13. *If G is n connected, $n \geq 2$ then its line graph $L(G)$ is also n-connected.*

Proof : Let G br n-connected and suppose that $\kappa(L(G)) < n$. Then removing $k < n$ nodes from $L(G)$ will produce either a disconnected trivial graph. Each node of $L(G)$ corresponds with an edge of G with two nodes of $L(G)$ adjacent if and only if the corresponding edges of G are incident. Thus by removing $k < n$

edges from G, we can produce either a disconnected or trivial graph, that is, $\kappa'(G) < n$. But by theorem 3.1, then implies $\kappa < \kappa' < n$, a contradiction. So $\kappa(L(G)) \geq n$, that is, $L(G)$ is n-connected.

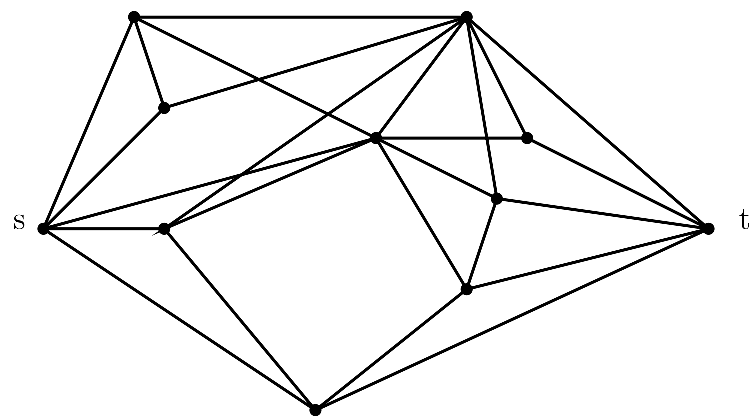
∎

Theorem 3.14. *For all integers a and b, $1 < a < b$, there is a graph G such that $\kappa(G) = a$ and $\kappa(L(G)) = b$.*

3.3 MENGER'S THEOREM

In 1927 Menger showed that the connectivity of a graph is related to the number of disjoint paths joining two nodes. Many of the variations and extensions of Menger which have since appeared have been graphical, and we discuss some of these here.

Let u and v be two nodes of a connected graph G. Two paths joining u and v are *disjoint* if they have no nodes other than u and v in common; they are *edge-disjoint* if they have no edges in common. A set of S nodes, edges, or both *separates* u and v if u and v are in different components of G_S. Clearly, no set of nodes can separate two adjacent nodes. In the following figure we display a graph with two nonadjacent nodes s and t which can be separated by removing three nodes but no fewer. The classical theorem of Menger guarantees the existence of three node-disjoint paths joining s and t.



A graph ilustrating Menger's theorem

Theorem 3.15. *The minimum number of nodes separating two non adjacent nodes s and t equals the maximum number of disjoint $s - t$ paths.*

Proof : It is clear that if k nodes separate s and t, then there can be no more than k disjoint paths joining s and t.

It remains to show that if it takes k nodes to separate s and t in G, there are k disjoint $s-t$ paths in G. This is certainly true if $k = 1$. Assume it is not true for some $k > 1$. Let h be the smallest such k, and let F be a graph with minimum number of nodes for which the theorem fails for h. We remove edh=ges from F until we obtain a graph G such that h nodes are required to separate s and t in G but for any edge e in G, only $h - 1$ nodes are required to separate s and t in $G - e$. We first investigate the properties of G.

By the definition of G, for any edge of G there exists a set $S(e)$ of $h - 1$ nodes which operates s and t in $G - e$. Now $G - S(e)$ contains atleast one $s - t$ path, since it takes h nodes to separate s and t in G.Each such $s - t$ path P must contain the edge $e = uv$ since P is not a path in $G - e$. So $u, v \notin S(e)$ and if $u \neq s, t$, then $S(e) \cup u$ separates s and t in G.

If there is a node w adjacent to both s and t in G,then $G - w$ requires $h - 1$ nodes to separate s and t so it has $h - 1$ disjoint $s - t$ paths. Replacing w, we have h disjoint $s - t$ paths in G. Thus we have shown.

1. No node is adjacent to both s and t in G.

Let W be any collection of h nodes separating s and t in G. An $s - W$ path is a path joining s with some $w_i \in W$ and containing no other node of W. Call all the collection of $s - W$ paths and $W - t$ paths P_s and P_t respectively. Then each $s - t$ path begins with a member of P_s and ends with a memeber of P_t, because every such paths in each collection and is some other node were in both an $s - W$ and $E - t$ path, then there would be an $s - t$ path containing no node of W. Finally, either $P_s - W = \{s\}$ or $P_t - W = \{t\}$ since, if not, then both P_s plis the edges $\{w_1 t, w_2 t, ...\}$ and P_t plus the edges $\{sw_1, sw_2, ...\}$ are h-connected graphs with fewer nodes than G in which s and t are nonadjacent, and therefore in each there are h disjoint $s - t$ paths. Combining the $s - W$ and $W - t$ portions of these paths, we can construct h disjoint $s - t$ paths in G, and thus we have a contradiction. Therefore we proved.

2. Any collection W of h nodes separating s and t is adjacent either to s or to t.

Now we can complete the proof. Let $P = \{s, w_1, s_2, ..., t\}$ be a shortest $s - t$ path in G and let $u_1, u_2 = z$. Note that by (1), $u_2 \neq t$. Form $S = \{v_1, v_2, ..., v_h - 1\}$ as above, separating s and t in $G - x$. By (1),$u_1 t \notin G$, so by (2), with $W = S(x) \cup \{u_1\}$, $sv_i \in G \forall i$. Thus by (1),$v_i t \notin G \forall i$. However, if we pick $W = S(x) \cup \{u_2\}$ instead, we have by (2) that $su_2 \in G$, contradicting our choice of P as shortest $s - t$ path.

∎

Corollary 3.16. *A graph G is n-connected if and only if every pair of nodes are joined by at least n node disjoint paths.*

Definition 3.17. *The local connectivity of two non adjacent nodes u and v of a graph is denoted by $\kappa(u, v)$ and is defined as the smallest number of nodes whose removal separates u and v. In these terms, Menger theorem asserts that for any two specific non adjacent nodes u and v, $\kappa(u, v) = \mu_0(u, v)$, the maximum number of node-disjoint paths joining u qand v.*

Theorem 3.18. *For any two nodes of a graph, the maximum number of edge disjoint paths joining them equals the minimum number of edges that separates them.*

Theorem 3.19. *A graph is n-edge connected if and only if every pair of nodes are joined by at least n edge-disjoint paths.*

Theorem 3.20. *For ant two disjoint nonempty sets of nodes V_1 and V_2, the maximum number of disjoint paths joining V_1 and V_2 is equal to the minimum number of nodes which separate V_1 and V_2.*

3.4 PROPERTIES OF N-CONNECTED GRAPHS

A collection of paths of a graph is an *independent set* if no two of them have anode in common. If each path in an independent set M of paths is an edge, then M is a *matching* of G. Thus a matching in a graph G is a set of edges of G, no two of which have a node in common.

Theorem 3.21. *A graph with at least $2n$ nodes is n-connected if and only if for any two disjoint sets V_1 and V_2 of n nodes each, there exist n independent paths joining tw sets of nodes.*

Proof : To show the sufficiency condition, we form a graph G' from G by adding two new nodes w_1 and w_2 with w_i adjacent to exactly the nodes $V_i = 1, 2$.
Since G is n connected, so is G' and hence there are n disjoint paths joining w_1 and w_2. The restrictions of these paths to G are clearly the n independent $V_1 - V_2$ paths.
To prove the other half, let S be the set of at least $n-1$ nodes which separates G into $_1$ and G_2. with node sets V_1' and V_2' respectively. Then since, $|V_1'| \geq 1, |V_2'| \geq 1$, and $|V_1'| \geq 1 + |V_2'| \geq 1 + |S| = |V| \geq 2n$, there is a partition of S into two disjoint subsets $|S_1|$ and $|S_2|$ such that $|V_1' \cup S_1| \geq n$ and.$|V_2' \cup S_2| \geq n$. Picking any n nodes each. Every path joining V_1 and V_2 must contain a node of S_1 and since we know that there are n independent $V_1 - V_2$ paths, we see that $|S| \geq n$, G is connected.

∎

Theorem 3.22. *In any graph, the maximum number of edge-disjoint cutsets of edges separating two nodes u and v is equal to the minimum number of edges in a path joining u and v that is the distance $d(u, v)$*

Definition 3.23. *Define a line of matrix as either a row or column. Every entry of a binary matrix is 0 or 1. In a binary matrix M, a collection of lines is said to cover all the unit entries of M if every 1 is in one of these lines. Two 1's of M are called independent if they are either in the same or row nor in the same direction.*
A node and an edge are said to cover each other if they are incident. A node cover of a graph G is a set of nodes which together covers all the edges of G. A matching that covers all the nodes of a grpah G is called perfect matching.

Theorem 3.24. *There exists a system of n representatives for a family of sets $S_1, S_2, ..., S_m$ if and only if the union of any k of these sets contain at least k elements, for all k to 1 to m.*

Theorem 3.25. *If G is bipartite, then the number of edges in the maximum matching equals the minimum number of nodes required to cover all the edges of G.*

Theorem 3.26. *Let G be a graph of order $p \geq 2$ whose node degrees d_i satisfy $d_1 \leq d_2 \leq .... \leq d_p$. Let n be an integer. $1 \leq n \leq p-1$. if,*

$$d_k \leq k + n - 2 \Rightarrow d_{p-n+1} \geq p - k$$

for each k such that $1 \leq k \leq \langle (p-n+1)/2 \rangle$, the G is connected.

Proof : Suppose G satisfies the conditions, but $\kappa(G) < n$. Then there exists a set S of at most n nodes whose removal disconnects G. Consider the smallest component H of $G - S$ and call its order k. Then $k \leq \langle (p-n+1)/2 \rangle$ and the largest degree of a node in H is at most $k + n - 2 < p - k$. Thus $d_k \leq k + n - 2$ and the hypothesis of the thorem then implies that $d_{p-n+1} \geq p - k$. Since each node in $V(G) - V(H) - S$ has degree atmost $p - k - 1$ and nodes in H also have degree less than $p - k$, only vertices in S have degree at least $p - k$. Now since $d_p \geq d_{p-1} \geq ... \geq d_{p-n+1} \geq p - k$, S contains at least n nodes, a contradiction. Thhus G is connected ∎

3.5 CIRCULANTS

In the first section, we noted that among all graphs with p nodes and q edges, $q \geq p - 1$, the maximum connectivity is $\langle 2q/p \rangle$ and this bound can always be attained. A chief reason for the importance of connectivity is its relation to the reliability and vulnerability of large scale computer and telecommunication networks.

Definition 3.27. *Maximum connectivity graphs plays an important role in the design of reliable networks. In this section, we discuss a class of graphs known as circulants which contains those graphs. For a given positive integer, let $n_1, n_2, ..., n_k$ be a sequence of integers where*

$$0 < n_1 < n_2 < ... < n_k < (p+1)/2$$

Then the circulant graph $C_p(n_1, n_2, ..., n_k)$ is the graph on p nodes $v_1, v_2, ..., v_p$ with vertex v_i adjacent to each vertex $v_{n_j(modp)}$. The values in n_i are called jump sizes.

Theorem 3.28. *The circulants $C_p(n_1, n_2, ..., n_k)$ satisfies $k < \delta$ if and only if for some proper divisor m of p, the number of distinct positive residues modulo m of the numbers $n_1, n_2, ..., n_k, p - n_k, ..., p - n_1$ is less than $min\{m - 1, \delta m/p\}$.*

Definition 3.29. *A regular graph with $\kappa = \delta$ for which the only minimum size disconnecting sets of nodes consists of the neighbourhoods of single nodes is called super-κ graph. Similarly, a regular graph with $\kappa' = \delta$ for which each minimum sized disconnecting sets of edges isolates a single node is called a super-κ' graph.*

Definition 3.30. *Let A be a set of all (p, q)-graphs G for which $\kappa'(G) = \kappa'$. A graph $G* \in A$ is κ'-optimal if it has the minimum number of disconnecting sets of edges of size κ' among all graph in A.*
A regular graph with $\kappa = \delta$ for which the only minimum size disconnecting sets of nodes consists of neighbourhoods of single nodes is called a super-κ graph. Similarly, a regular graph with $\kappa' = \delta$ for which minimum sized disconnecting sets of edges isolates a single node is called a super-κ' graph.

Definition 3.31. *When designing a communication network, one not only wants ti maximize the connectivity and edge-connectivity but also to minimize the diameter as well as the number of the edges. By minimizing the diameter, transmission times are kept small and the possibility of distortion due to a weak signal is avoided. Minimising the number of edges will keep down the cost of building the network. Of course, one cannot have everything that is, in general one cannot simultaneously maximise* κ *and* κ' *while maximising* $|E|$ *and* $d(G)$.

Definition 3.32. *Two elements of a lattices are incomparable if neither dominates the other. By a chain in a lattices is meant a downward path from upper element to a lower element in the Hasse diagram of lattice.*

A graph G *is* k*-critically* n*-connected if for all* $S \subset V(G)$ *with* $|S| \leq k$*, we have* $\kappa(G - S) = n - |S|$.

The edge persistence of a graph is the minimum number of edges that must be removed to increase its diameter.

Chapter 4

EXTREMAL DISTANCE PROBLEMS

4.1 INTRODUCTION

Let f be a real valued function whose domain is the set of all graphs and let$P$ be any graphical property. As with the topics discussed in the previous chapters, extremal graph theory is an area on which whole book could be written. In this chapter we focus our attention on extremal problems relating to radius, diameter,and long paths and cycles in graphs.

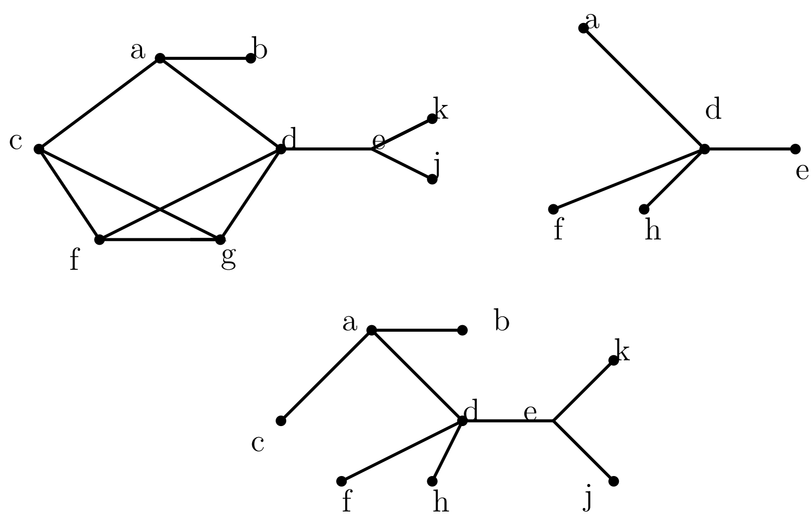
4.2 RADIUS

In chapter 2 we defined the radius $r(G)$ of a connected graph G as the minimum eccentricity of its nodes and the center $C(G)$ as the set of all nodes v

with $e(v) = r(G)$. Center plays an important role in a number of facility location problems. Since $r(G)$ is a real valued function, in fact positive integer valued, the radius could play the role of the function f in the extremal graph paradigm. However in many extremal problem, $r(G)$ instead plays a role in property P.

Definition 4.1. *For any connected graph G, it is easy to generate a spanning tree T of G for which the distances from a fixed node v are preserved. One simply uses the well known breath first search algorithm with roor v. This algorithm begins at a node v and branches out to its neighbours u, including the edges uv in the tree. Next, the edges joining those nodes at distance one from v with nodes at a distance two from v are indicated so as not to form any cycles. this process continues until a spanning tree is formed. The process is illustrated in the following figure where the central node ,the spanning tree T will have the same raduis as G. Such a tree is called radius preserving spanning tree.*

If one removes an edge from a graph G, it is clear that the radius may increase or stay the same, but it certainly could not decrease. A graph G is called radius minimal if $r(G - e) > r(G)$ for every edge e in G.



A breath first search at a node

Theorem 4.2. *A nontrivial graph G is radius minimal if and only if G is a tree.*

Proof : If G is a tree, then clearly G is radius minimal since the removal of any edge will disconnect G, resulting in an infinite radius.

If G is radius-minimal, then $r(G)$ must be finite so G is connected. Assume that G is connected and not a tree. Then G has a radius preserving spanning tree T which necessarily has fewer edges than G. Thus it is possible to remove an edge from G without decreasing the radius. Hence a radius-minimal graph must be a tree.

Next we consider graphs whose radius is altered by the removal of any node. A nontrivial graph G is called *r-critical*, or briefly *r-critical*, if for every node v in G, $r(G - v) \neq r(G)$. Every even path P_{2n} is critical. By removing an endnode of P_{2n}, the radius decreases by one, but removing an internal node of P_{2n} makes the radius unbounded. It is a simple observation that if G is an r-critical graph and v is one of its nodes, then $r(G - v) < r(G)$ if and only if v is a peripheral node, and in this case $r(G-v) = r(G) - 1$ ∎

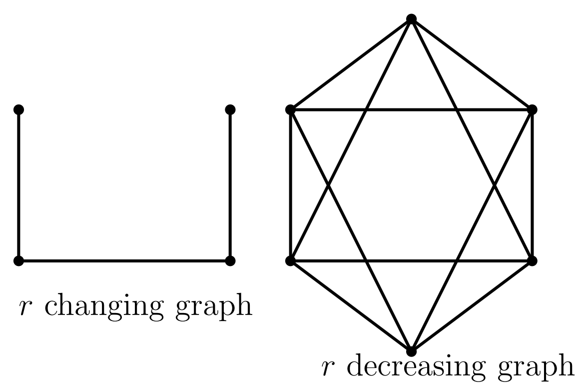
The class of *r-critical graphs* can be partitioned into three sets:

r-decreasing graphs for which $r(G-v) = r(G) - 1 \forall v$;

r-increasing graphs for which $r(G-v) > r(G) - 1 \forall v$;

r-changing graphs which comprise all other r-critical graphs.

Thus each r-changing graph contains at least one node v for which $r(G-v) > r(G)$ and one node u for which $r(G-u) < r(G)$. The following figure gives an example of r-changing graph and r-decreasing graph.



Theorem 4.3. *Every connected r-critical graph G is either r-decreasing or consists of an r-decreasing subgraph H and endpaths so that one endpath of length $r(G) - r(H)$ is joined to each node of H.*

Corollary 4.4. *There are no r increasing graphs.*

Theorem 4.5. *A graph.0 G of radius 2 is r-critical if and only if it is either the path P_4 or the complete multipartite graph $K(2, 2, .., 2)$ with $n \geq 2$ parts.*

4.3 SMALL DIAMETER

Definition 4.6. *One of the earlier results for graphs with small diameter concerns a special class of graphs called moore graphs. The distance degree sequence of a node v is $dds(v) = (d_0(v), d_1(v), ..., d_{e(v)}(v))$, where $d_i(v)$ is the number of nodes at distance i from v. If every node of G has the same distance degree sequence then G is called distance degree regular. A Moore graph is a DDR graph, so it is self-centered and self-median. Hence any node could play the role of the peripheral node v. The girth of a graph is the length of the any shortest cycle. These are special examples of a class of highly symmetric graphs called cages. Deleting an edge from a graph may cause its diameter to increase or stay the same, but it cannot decrease. A graph G is diameter-minimal if for all edges $e \in G$, $d(G - e) > d(G)$. Any edge that can be removed from G without affecting the diameter is called superfluous. Thus diameter-minimal graphs are the graphs with no superfluous edges. Suppose that G has diameter 2. Then every superfluous edge $e = uv$ is contained in a triangle, since otherwise removal of e would make $d(G) \geq d(u, v) \geq 3$.*

Theorem 4.7. *Every graph G can be imbedded as an induced subgraph in a diameter-minimal graph of diameter 2.*

Proof : Label the nodes of G by $v_1, v_2, ..., v_p$. Next, add new nodes $w, x, u_1, u_2, ..., u_{p+1}$ and edges v_iu_i, wu_{p+1}, and xu_i. Finally, for each pair of distinct non adjacent nodes v_i, v_j insert the edge u, u_j. It is easy to verify that the resulting graph is diameter-minimal, has diameter 2 and has G induced subgraph.

∎

Definition 4.8. *A graph G is diameter-critical if $d(G-r) \neq d(G)$ for every node $v \in G$. Note that in a diameter-critical graph G, some nodes of G may cause the diameter to increase when removed while others can cause to decrease. For example, if G consists of C_7 with a endedge attached, then removal of the node of degree one would cause the diameter to decrease, whereas the removal of any other node increases the diameter.*

Theorem 4.9. *Let G be a connected graph and let X be the set of all nodes x for which $d(G-x) < d(G)$. Then $|X| \leq 2$ and $d(G-2) \leq d(G-X) \leq d(G)-1$ for $X \neq \emptyset$.*

Proof : If $x \in X$ then $d_G(u,v) \leq d(G-x) < d(G) \forall u, v \in G-x$. Hence if $d(y,w) = d(G)$ for some pair of nodes $y, w \in G$, then y or w must be x. So each node in X is peripheral. Any pair of nodes $x, x' \in X$ must be antipodal nodes or else removing one of them could not decrease the diameter. But then if X contains three or more nodes, removal of one node say $x*$ would still leave an antipodal pair of nodes at distance $d(G)$ from one another in $G - x*$, contradicting the fact that $d(G-x*) < d(G)$. Thus $|X| \leq 2$.

∎

Definition 4.10. *A graph G is d-increasing id $d(G-v) > d(G) \forall v \in G$ and d-decreasing if $d(G-v) < d(G) \forall v \in G$*

Theorem 4.11. *Let G be a diameter-critical graph of diameter $d \leq 3$ that is not d-decreasing. Then G is a path of length d.*

Trees of Small Diameter: Although there are infinitely many trees of diameter at most there, they are easy to describe. They are the graphs K_1, K_2, the stars $K_{1,n}$ and the double stars $S_{m,n} = \mathrm{K}_m + K_1 + K_1 + \mathrm{K}_n$. These trees have been used in various situation in the literature, perhaps the most interesting of which in decomposition problems or in packing problems.
The slight difference between packing problems and decomposition problems is one of generality. In a typical packing problems, one begins with a well known class of graphs like K_p and wants to determine whether to colour the edges of the graph so that the colour determines some fixed set of trees of small diameter.

Theorem 4.12. *Suppose $T_2, T_3, ..., T_p$ are trees such that T_i, has order i and $d(T_i) \leq 3$. Then $T_2, T_3, ..., T_p$ can be packed into K_p.*

Definition 4.13. *Recall that a graph is non-separable if it is connected and has no cutnodes. Such a graph has only one block, and for that reason, the graph itself is often called a block. A block G is minimal $G - e$ is not a block for each edge $e \in G$*

Theorem 4.14. *The minimal blocks of diameter 2 are as follows;*

1. $K_{2,p-2}$ with $p \geq 4$

2. The graph formed from the double star by adding a node v and joining v to *each endnode of $S_{m,n}$.*

4.4 DIAMETER

Theorem 4.15. *For any graph G and integer $k \geq 4$, there exists a diameter-critical graph H with diameter k containing G as an induced subgraph.*

Proof : Begin by adding additional node $v_{1,0}$ to G and joining $v_{1,0}$ to each node of G to form graph F. Label the nodes of G by $v_{1,t}, 1 \leq t \leq p$. Next take a copy of $\bar{\text{F}}$ with nodes $v_{k-1,t}, 0 \leq t \leq p$. Add additional nodes $v_{i,t}, 2 \leq i \leq k-2, 0 \leq t \leq p$ and join each pair of nodes $v_{1,t}$and $v_{k-1,t}$ by the path $v_1, v|_2, ..., v_{k-1}$. Finally add two more nodes u and w join u to w and to all nodes.
It is easy to check the resulting graph H has diameter d and it clearly contains G as an induced subgraph. It remains to verify that H is diameter-critical. Graph $H - w$ has diameter $d - 1$ and $d(H - u) = \infty$. Upon removal $v_{i,j}$ with $2 \leq i \leq k-1$, one finds the distance $d(w, v_{i,j}) > k$ in $G - v_{i,j} > k$. Removing $v_{i,j}$ yields the distance $d(v_{2,j}, x) > k$ in $G - v_{1,j}$ for $x \in N_F(v_{i,j})$. Hence each node alters the diameter. So H is diameter-critical.

∎

Theorem 4.16. *Let $G \neq K_2$ be a diameter-critical graph on p nodes. Then $\delta(G) \leq \langle (p - d + 1)/2 \rangle$.*

Theorem 4.17. *For any (p, q)-graph of diameter d, we have $q \leq d + \frac{1}{2}(p - d +$*

$1)(p-d+4)$.

Definition 4.18. *A connected noncomplete graph G is n-geodetically connected if the removal of at least n nodes are required to increase the distance between every pair of nonadjacent nodes. The geodetic connectivity is the maximum n such that G is n-geodetically connected. If G is n-geodetically connected, then it is obviously n-connected, but the converse is not true. For example, the graph $K_4 + \bar{K}_2$ is 3-connected , but only 2-geodetically connected. Note that every graph with geodetic connectivity equal to one is diameter-critical.*

Theorem 4.19. *The following assertions are equivalent for a graph G:*
1. G is n-geodetically connected.
2. G is connected and every two nodes at distance two from one another are joined by at least n geodesics.
3. For every pair of distinct nonadjacent nodes u and v any set of $m \leq n$ disjoint $u-v$ geodesics is contained in a set of n disjoint $u-v$ geodesics.
4. For any $n+1$ distinct nodes $v_0, v_1, ..., v_n$, G contains disjoint $v_0 - v_1, v_1 - v_2, ..., v_0 - v_n$ geodesics.

Among all invariants studied in connection with diameter extremal problems except maximal degree, connectivity has played an important role. It has generated the most interest in problems involving regular graphs. A minimum (t, k, n)-graph is an n-regular graph G of minimum order with $\kappa(G) = k$ and diameter $d(G) = t$.

Theorem 4.20. *A diameter maximal graph G of diameter $t \geq 4$, connectivity n and order p having the maximum number of edges has the form*

$$K_1 + K_n + K_{a2} + ... + K_{at-2} + K_n + K_1$$

with $a_i = n$ for each i except possibly one or two consecutive a_i for which $a_i > n$.

Theorem 4.21. *A diameter maximal graph G of diameter $t \geq 6$, connectivity n and order p having the maximum number of edges has the form*

$$K_1 + K_n + K_{a2} + ... + K_{at-2} + K_n + K_1$$

where every triple (a_{i-1}, a_i, a_{i+1}),$3 \leq i \leq t-3$, except possibly one, contains exactly $n-1$ nodes. The exceptional triple is either (a_2, a_3, a_4) or $(a_{t-4}, a_{t-3}, a_{t-2})$

4.5 LONG PATHS AND LONG CYCLES

Definition 4.22. *Recall that a trial is a walk in which no edge appears more than once. Thus in a trial, nodes can be revisited, but edges cannot. In a path, neither nodes nor edges may be repaeated. The trial number $tr(G)$ of a grpah G is the maximum length of a trial in it.*

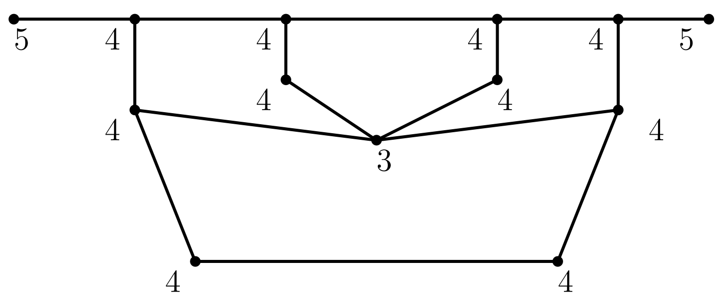
Theorem 4.23. *The maximum trial number among all graphs on p nodes and q edges is*

$$tr(G) \le q podd\ or\ q \le \binom{n}{2} - \frac{n}{2} + 1 \binom{n}{2} - \frac{n}{2} + 1\ otherwise$$

Proof : If p is odd then K_p is eulerian and the p nodes of K_p along with the first q edges of any eulerian trial of K_p form a (p, q)-graph G with $tr(G) = q$. If p is even, then $K_p - (\frac{p}{2} - 1)K_2$ has a spanning trial that includes every edge. Hence for $q \le \binom{n}{2} - \frac{n}{2} + 1$, the first q edges in such a trail again produces a (p, q)-graph G withg $tr(G) = q$. On the other hand, if p is even, $tr(G)$ can be no larger than $\binom{n}{2} - \frac{n}{2} + 1$ since any subgraph H formed by the edges of a trial in G has at most two nodes of degree $p - 1$. So

$$tr(g) = |E(H)| \le \frac{1}{2}(2(n-1) + (n-2)^2) = \binom{n}{2} - \frac{n}{2} + 1$$

∎



A graph with all diametral paths avoiding center

Theorem 4.24. *Suppose that all diametral paths of G avoid the center then,*

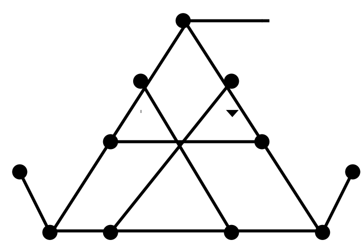
$$r(G) + 2 \le d(G) \le 2r(G) - 1$$

Definition 4.25. *A detour in a graph G is a path of maximum length and the length of such a path is called detour number $dn(G)$.*

Let $\omega(H)$ denote the number of components in a graph H. A graph G is t-tough *if each subset* $S \subset V(G)$ *with* $\omega(G-S) > 1$ *satisfies*

$$|S|/\omega(G-S) \geq t$$

The toughness of a graph is the maximum value of t for which it is t-tough.



A graph with only 12 nodes having
a detur avoiding each given node

Theorem 4.26. *If G is a connected graph on p nodes then,*

$$dn(G) \geq min\{p-1, 2\delta(G)\}$$

Theorem 4.27. *If G is 1-tough with order* $p \geq 3$ *such that* $degu + degv \geq k$ *for all distict adjacent nodes* u, v*, then the circumference of G is atleast* $min\{p, k+2\}$.

Theorem 4.28. *Suppose that G is 1-tough with order* $p \geq 3$ *such that* $degu + degv + degw \geq k$ *for all independent triples of nodes* u, v, w*.Then the circumference of G is at least* $min\{p, p/2 + k/3\}$.

Theorem 4.29. *let G be 2-connected graph with order p and let k be an integer with* $3 \leq k \leq p$. *If for all pairs of nodes* u, v *at distance two from one another*

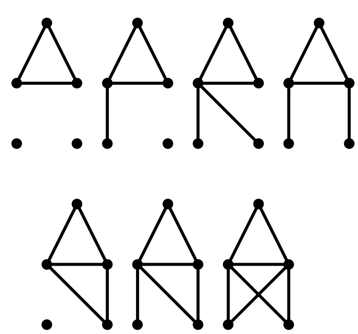
$max\{degu, degv\} \geq k/2$, *then the circumference of* G *is at least* k.

Theorem 4.30. *For a graph* G, *if* $k \geq 3$ *is an odd positive integer and let*

$$\delta(G) \geq \frac{(k+1)^k}{k} \text{ or } q \geq \frac{(k+1)^k - k - 1}{k}$$

then for every natural number t, G *contains a cycle of length* $tmodK$.

Theorem 4.31. *Let* $k \geq 3$ *be a fixed positive integer. IF* G *is a 2-connected non-bipartite graph on* p *nodes with* $\omega(G) \geq 2p/(k+2)$, *then for* p *large either* G *contains the cycle* C_k *or* G *is isomorphic to the graph obtained from* C_{k+2} *by replacing each of its nodes.*



Induced subgraphs of G with $g(G) = 3$

Definition 4.32. *Let* G *be a graph with diameter* $d(G) = n$. *Then its clique graph* $K(G)$ *has diameter* $n+1$ *if and only if* G *has cliques* C *and* D *such that* $d(x, y) = n$ *for every pair of nodes* $x \in C$ *and* $y \in D$.

A graph G *is diameter edge invariant if its diameter is unchanged by the deletion* *of an edge, that is,* $d(G - c) = d(G) \forall c \in G$.

Chapter 5

DISTANCE SEQUENCES

5.1 INTRODUCTION

A *sequence for graph* is simply an invariant which consists of a list of numbers rather than a single number. The advantage of studying and using a sequence is that it is often nearly as easy to calculate as a single numerical invariant yet it carries far more information about the graph it represents. In this chapter, we discuss a number of distance related sequences for a graph, display their relation to one another a well as to various concepts in graph theory.

5.2 THE ECCENTRIC SEQUENCE

A sequence is *graphical* if there is a graph which realises S. Before discussing the eccentric sequence, we present results on the only graph sequence which predated it.

Theorem 5.1. *The sequence* $D = (d_1, d_2, ..., d_p)$ *with*

$$p - 1 \geq d_1 \geq d_2 \geq ... \geq d_p$$

is a graphical sequence iff the modified sequence

$$D' = (d_2 - 1, d_3 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_p)$$

is a graphical degree sequence.

Proof : If D' is a graphical degree sequence, then so is D, since from a graph with degree sequence D' one can construct a graph with degree sequence D by adding a new node adjacent to the nodes having degrees $d_2-1, d_3-1, ..., d_{d+1}-1$. Now let G be a graph with degree sequence D. If anode of degree d_1 is adjacent to nodes of degree d_k for $k = 2$ to $d_1 + 1$, then the removal of this node results in graph of degree sequence D'.
Suppose that G has no such node. We will show that from G one can always get another graph with degree sequence D having such a node. We assume that the nodes in G are labelled so that $degv_i = d_i$ and that v_1 is a node of degree d_1 for which the sum of the degrees of the adjacent node is maximum. Then there are nodes v_i and v_j with $d_i < d_j$ such that v_1v_j is an edge but v_1v_i with $d_i > d_j$ such that v_1v_j is an edge but not to v_j. Removal of a the edges v_1v_j and v_kv_i and the addition of v_1v_i and v_kr_j results in another graph with degree sequence D. But in this new graph, the sum of the degrees od the nodes adjacent to v_1 is greater than before since v_1 is now adjacent to v_i rather than v_j. By repeating

this edge-switching process a finite number of times, we obtain a graph with degree sequences D in which v_1 has the desired property.

∎

Algorithm:

The sequence $D = (d_1, d_2, ..., d_p)$ with $p - 1 \geq d_1 \geq d_2 \geq .. \geq d_p$ is a graphical degree sequence iff the following procedure results in a sequence with every term zero.

1. Determine the modified sequence in D' as described in above theorem.

2. Reorder the terms of D' so that they are in nonincreasing order, and call the resulting sequence D_1

3. Determine the modified sequence D^n of D_1 as in step 1 and reorder D'' as in step 2 call the recorded sequence D_2.

If a sequence at an intermediate stage of the algorithm is known to be agraphcial degree sequence stop, since D itself is then established be one also.To illustrate we test the sequence

$D = (5, 5, 3, 3, 2, 2, 2)$

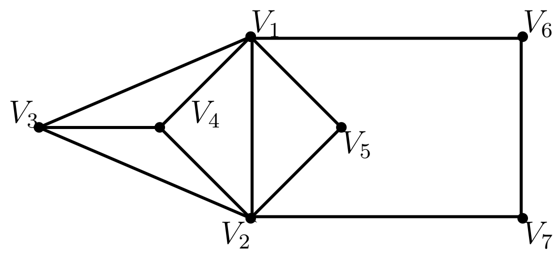
$D' = (4, 2, 2, 1, 1, 2)$

$D_1 = (4, 2, 2, 2, 1, 1)$

$D'' = (1, 1, 1, 0, 1)$

$D_2 = (1, 1, 1, 1, 0)$

Clearly, D_2 is a graphical degree sequence si D is also.



An example for above algorithm

Definition 5.2. *The eccentric sequence of a connected graph G is a list of the eccentricities of its nodes in non decreasing order. Since there are often many nodes having the same eccentricity, we will simplify the sequence by listing as*

$$e_1^{m_1}, e_2^{m_2}, ..., e_k^{m_k}$$

where e_i are eccentricities $(e_i < e_{i+1})$ and m_i is the multiplicity of e_i.
Some simple observations about the values of e_i and m_i for a non trivial connected graph are as follows:
1. Since for each pair of adjacent nodes u, v and ant third node w,$|d(u, w) - d(v, w)| \leq 1$, it follows that the e_i are consecutive positive integers.
2. $e_1 = r(G)$ and $e_k = d(G)$,so $1 \leq e_i \leq p - 1$
3. Since there must be apair of diametral nodes, $m_k \geq 2$.
4. Since the diameter is at most twice the radius, $e_k \leq 2e_1$.

Theorem 5.3. *For $p \geq 2, m_i \geq 2$ except possibly for m_1.*

Proof : Since the e_i are consecutive positive integers, there is at least one node of eccentricity t for each integer $t, e_1 < tleqe_k$. Let u be anode with eccentricity

$t > e_1$ in G, and let v be an eccentric node of u. Then $e(v) \geq t$. For a central node w, let P be a $v-w$ geodesic. Sine $e(w) = e_1, d(v, w) \leq e_1$. Since the eccentricities of adjacent nodes can differ by at most one, and $e(w) = e_1 < t \leq e(v)$, some node x on P has eccentricity t. Since $d(u, v) = t > e_! \geq d(x, v)$, node x must be distinct from u. Thus there are at least two nodes with eccentricity t.

∎

Lemma 5.4. *For all positive integers r and d satisfying $r \leq d \leq 2r - 2$, there exist graphs with radius r and diameter d. The minimum order of such a graph is $r + d$. There are exactly $\langle (d - r)/2 \rangle + 1$ non isomorphci graphs of order $r + d$, radius r and diameter d. Each graph consists of a path $u_0, u_2, ..., u_d$ and a path $u_s, v_1, v_2, ..., v_{r-1}, v_{s+r}$ with only the nodes u_s and u_{s+r} in common.*

Definition 5.5. *An eccentric sequence is minimal if it has no proper eccentric subsequences with the same number of distinct eccentricities.*

Theorem 5.6. *A sequence S of positive integers is eccentric if and only if some subsequence T of S is eccentric.*

5.3 DISTANCE SEQUENCES

The distance degree sequence consists of a collection of sequences. For a node v in a connected graph G, let $d_i(v)$ be the number of nodes at distance i from v. The *distance degree sequence of node v* is

$$dds(v) = (d_0(v), d_1(v), d_2(v), ..., d_{e(v)}(v))$$

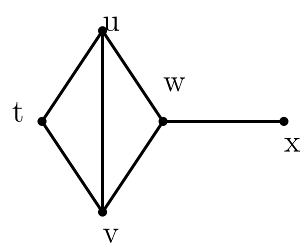
Note the following:

1. $d_0(v) = 1 \forall v; d_1(v) = deg v$.

2. The length of sequence $dds(v)$ is one more than the eccentricity of v.

3. $\sum d_i(v) = p$.

The *distance degree sequence dds(G) of a graph* G consists of the degree sequence arranged in an numerical order. If a particular dds appears k times, we list it once with k as an exponent to indicate the multiplicity. For example in the following figure, $dds(t) = (1, 2, 1, 1)$, $dds(w) = (1, 3, 1)$ and $dds(G) = (1, 1, 2, 1; (1, 2, 1, 1); (1, 3, 1)^3)$



A graph to illustrate degree sequence

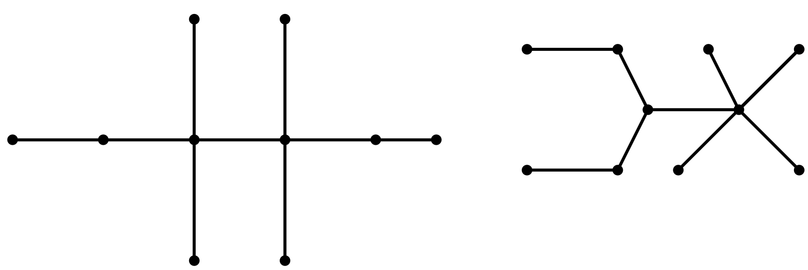
The *distance degree regular(DDR) graphs* is the graphs in which all nodes have same distance dgree sequence.Thus DDR graph has the property that $dds(G) = ((dds(v))^p)$, where v is any node in G. A DDR graph G is necessarily regular since $d_1(v) = d_1(w)$ for any two nodes v and w in G. However, the converse is not true.

A connected graph G is *distance degree injective (DDI)* if the distance degree sequences of its nodes are all distinct. As opposed to DDR graphs, these graphs are completely assymmetric. Indeed, all DDi graphs have identity automorphism group. There are DDR graphs of ever order and every diamter because of the K_n and C_n.

Theorem 5.7. *Every regular graph containing a cutnode is not DDR.*

Proof : Let G be a connected regular graph with a cutnode v, and let G_1 and G_2 be components of $G - v$. Suppose that an eccentric node of v in G lies in G_2, and let x be a neighbour of v which lies in G_1. Then the eccentricity of x within G is greater than the eccentricity of v in G. Thus $dds(x) \neq dds(v)$.

∎



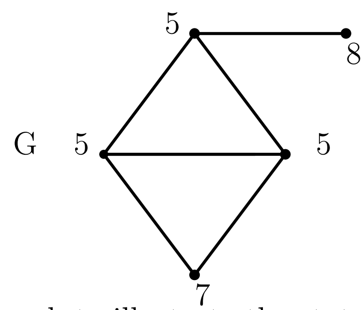
A pair of graphs with the same status sequence

Theorem 5.8. *If G is non trivial graph for which both G and $\bar{G}$ are DDI, then both G and $\bar{G}$ have diameter 3.*

Proof : The only graphs with diameter 1 are the complete graphs and for $p > 1$, they are DDR but not DDI. Next, suppose that G is a graph with diameter 2. Then the distance degree sequence of any nodes of G have the same degree. Thus no DDI graph has diameter 2. Finally if $d(G) > 3$, then $d(\bar{\text{G}}) leq 2$. Thus aDDI graph cannot have diameter greater than three if its complement is also a DDI graph.

∎

Definition 5.9. *The status sequence ss(G) of a connected graph G is the list of its status values arranged in non decreasing order. The following graph has status sequence* $(5^3, 7, 8)$.



A graph to illustrate the status sequence

The relationship between status sequence and median problems is analogous to the relationship between eccentric sequences and center problems. There are several properties which distinguish status sequences from eccentric sequences.

1. The status values need not be consecutive integers.

2. There need not be two nodes having maximum status.

3. $ss(G)$ is derivable from $dds(G)$; For the sequence $dds(v) = (d_0(v), d_1(v), ..., d_{e(v)}(v))$, we have

$$s(v) = \sum_{i=1}^{e(v)} i - d_i(v)$$

It is easy to see that their distance degree sequences are different because their diameters differ as do their degree sequences. Obviously, all statuses of nodes in

K_p are equal, as are those in C_p. Thus these graphs are self-median so are al DDR graphs. In searching for self-median graphs, one usually finds that the graph is regular. Self-median graphs are also called *status sequence regular graphs*. The other extreme from self-median graphs, we may consider graphs G for which all of the terms of $ss(G)$ are distinct. We call these graphs as *status injective SI*.

5.4 THE DISTANCE DISTRIBUTION

Let D_1 be the number of pair of nodes at distance i from one another in the connected graph G with diameter d. Then the *distance distribution* of G is the sequence

$$dd(G) = (D_1, D_2, ..., D_d)$$

Obviously, $dd(G)$ is obtained at once from $dds(G)$ as $2D_i = \sum d_i(v)$ with the sum taken over all nodes v of G. Also, note that $D_1 = q$, the number of edges in G.

Although $dd(G)$ can be derived from $dds(G)$, it still contains a wealth of information and deserves a separate treatment. In fact, in certain problems,$dds(G)$ contains too much information and is cumbersome to work with whereas $dd(G)$ is ideal for the problem. However, remembering that $dd(G)$ can be derived from $dds(G)$ is useful.

Theorem 5.10. *If G is a connected graph on p nodes, then $D_1 + D_2 \geq 2p - 3$.*

Theorem 5.11. *When G is a tree, D_2 is given by the degree sequence*

$$\binom{D_2=\sum_{i=1}^{p} degv_i}{2}$$

Proof : Let $N(v)$ be the set of neighbours of v. Each pair of nodes in $N(v)$ are joined by a unique path, which necessarily passes through v. The term $\binom{degv_i}{2}$ counts the number of pair of nodes that are at distance two, via v_1 from each other. by summing over all v_i, the result follows.

For a connected graph G, let $s_k(G)$ be the kth partial sum of $dd(G)$ that is,

$$s_k(G) = \sum_{i=1}^{k} D_i$$

∎

Theorem 5.12. *Let T be any tree on p nodes. Then $s_k(T) \geq s_k(P_p)$, and the equality holds for all k iff $T = P_p$*

Proof : The result is clear for small p by induction process. Assume it is true for all $t < p$ and let T be any tree on p nodes. Let $d(G) = d$ and $e(v) = d$, that is,v is the end node of a diametral path in T. By the inductive hypothesis, $s_k(T - v) \geq s_k(P_{p-1}) \forall k$. By attaching extra node to an end node of P_{p-1} by exactly 1. By reattaching v to $T - v$, we increase each D_i of $T - v$ by atleast 1. By D_d we have accumulated an increase of $p - 1$, since each node of T can be reached from v by a path of length d or less. Thus, $s_k(T) \geq s_k(P_p)$ for each k.

If $T \neq P_p$ then $d(P_p) = p - 1$ and $d(T) < p - 1$. Therefore,

$$s_{p-2}(T) = s_{p-1}(T) = s_{p-1}(P_p) > s_{p-2}(P_p)$$

Hence the equality holds for all k if and only if $T = P_p$

∎

Definition 5.13. *If a graph has a path of length k, then it has a path of each smaller length. This may lead one to feel that $dd(G)$ must be a non increasing sequence. Not only is this not in case, but $dd(G)$ must be non increasing sequence. Not only is this not the case, but $dd(G)$ need not evn be unimodal. A sequence S_i is unimodal if there is some k for which $S_i \leq S_{i+1}$ for $i < k$ and $S_j \geq S_{j+1}$ for $j \geq k$.*

A distance distribution is uniform if $D_i = D_j \forall i$ and j. Thus, if G has diameter d and $dd(G)$ is uniform, then $\binom{D_i = p}{2/d}$.

Theorem 5.14. *1. If $\binom{p}{2}$ is even, there exist at least two uniform distance distributions for p.*

2. if $\binom{p}{2}$ is divisible by 3, there are at least two graphs of order p with uniform distance distributions, except for $p = 3$ and 6.

3. If $p = 12k + 2$ and $\binom{p}{2}$ is the product of two primes, then K_n is the unique graph of order p with a uniform distance distribution.

Definition 5.15. *The mean distance $\mu_D(G)$ of a connected graph G is the average of the distances between pairs of nodes in G. Of copurse, $\mu_D(G)$ can be calculated from $dd(G)$:*

$$\binom{\mu_D(G)=\sum_{i=1}^{d} iD_i/p}{2}$$

Since the mean distance and the radius are both measures the central tendency of a graph. For such graphs we obatin rather a nioe raltionship between r *and* D_i *from* $dd(G)$*. We know that,*

$$\binom{\sum_{i=1}^{d} D_i=p}{2}$$

Knowing this and setting $\mu_D(G) = r(G)$*, we obatin our result.*

Theorem 5.16. *If* $r(G) = d(G) \geq 2$*, then* $\mu_D(G) \neq r(G)$*.*

Theorem 5.17. *If* $d(G) = 3$*, then* $\mu_D(G) = r(G)$ *if and only if* $r(G) = 2$ *and* $D_1(G) = D_3(G)$*.*

Proof : If $d = 3$, then $r = 2$ or 3. Suppose that $\mu_d(G) = r(G)$. Thus by the above theorem, r must be 2. Substituting in relation between r and D_i we get, $\binom{D_2+2D_3=p}{2}$. But $\binom{D_1+D_2+D_3=p}{2}$. Thus $D_1 = D_3$

If $r(G) = 2$ and $D_1 = D_3$, then,

$\binom{\mu_D(G)=(D_1+D_2+D_3)/p}{2}$

$= (2D_1 + 2D_2 + 2D_3)/(D_1 + D_2 + D_3) = 2 = r(G)$

∎

Corollary 5.18. *The only tree* T *of diameter 3 with* $\mu_D(T) = r(T)$ *is the tree having degree sequence (4,3,1,1,1,1,1).*

Definition 5.19. *A caterpillar is a tree T having a diametral path incident with every edge of T. A tree which we call the double starred path $P_{a,b,c}$ is the graph formed from P_a by attaching b pendant edges at one end and c pendant edges at the other. By joining various pairs of end nodes in such graphs, we were able to show that there are graphs for every diamter $d \neq 2$ for which $\mu_D(G) = r(G)$.*
The edge density $\rho(G)$ of (p,q)-graph G is $\binom{q/p}{2}$. For each rational number $t > 1$ there are infinitely many graphs G with $\mu_D(G) = t$.
Three nodes of a graph are said to be collinear if they can be labelled u, v, w so that

$$d(u,v) + d(v,w) = d(u,w)$$

The collinearity ratio $cr(G)$ of a graph G is the proportion of collinear triples of nodes in G. Thus

$cr(G) = \textit{number of collinear triples}_{\overline{p3}}$

We define an *equilateral triangle* in a graph G as a set $\{v_i, v_j, v_k\}$ of three nodes such that all three distances d_{ij}, d_{ik}, d_{jk} are finite and equal. They characterised the connected graphs having no equilateral triangles. We note that inserting additional edges into a a graph may increase, decrease nor not affect the value of $cr(G)$.
A connected graph G is *geodetic* if any twonodes u, v are joined by precisely one path of length $d(u,v)$.

Theorem 5.20. *For any connected graph G on $p \geq 3$ nodes, we have*

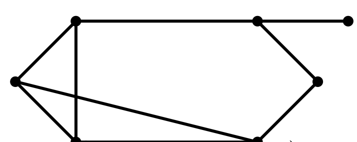
$$cr(G) \geq 3(\mu_D(G) - 1)/(p - 2)$$

Theorem 5.21. *If G is geodetic, then*

$$\mu_D(G) = \frac{(p+1)}{3} - \frac{r(G)}{p2}$$

5.5 PATH SEQUENCES

Chronologically, this was the second graphical sequence to be studied. Let l_i be the number of apirs of nodes joined by a path of length i. Capobinaco defined the *path length distribution(pld)* of a connected graph G as the sequence $(l_1, l_2, ..., l_{p-1})$.



(9,13,15,15,15,3)
A graph and its path length distribution

Theorem 5.22. *For every $p \geq 9$ there are pairs of trees on p nodes with the same pld. Moreover, for any integer n, one can contruct n trees having the same pld.*

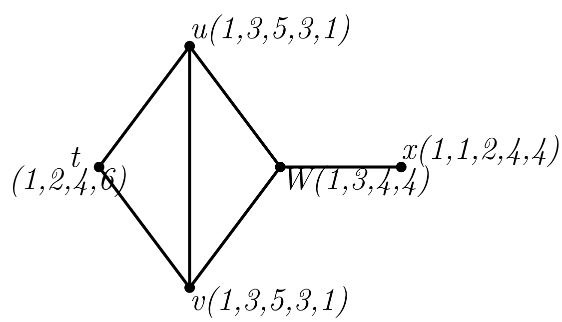
For any trees T, the sequence $pld(T)$ will end with a string of zeros beyond l_d. There is a unique path between any pair of nodes in a tree. Thus for any tree T

the first d terms of $pld(T)$ corresponds precisely to $dd(T)$.

Corollary 5.23. *The distance distribution distinguishes non isomorphic trees only for $p \leq 8$.*

Definition 5.24. *For each node v in a connected graph G, let $p_i(v)$ be the number of paths of length i beginning at v. Then define the path degree sequence of v as*

$$pds(v) = (p_0(v), p_1(v), p_2(v), ..., p_{p-1}(v))$$



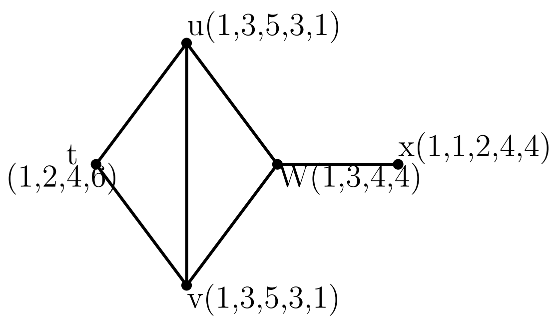
The path degree sequence of the nodes of a graph

The sequences $pds(v)$ generally end with a string of zeros, so we terminate the sequence at the last nonzero term.

1. $p_0(v) = 1 \forall v$
2. $p_1(v) = degv$
3. if G is a tree, then $pds(v) = dds(v)$.

The *path degree sequence* $pds(G)$ of a graph G consists of the collection of sequences $pds(v)$ of its nodes, listed in numerical order. If a particular pds appears k times, we list it once with k as an exponent to indicate the multiplicity. For example in the above figure, $pds(t) = \{1, 2, 4, 6\}$, $pds(w) = \{1, 3, 4, 4\}$ and $pds(G) = ((1, 1, 2, 4, 4,), (1, 2, 4, 6); (1, 3, 4, 4); (1, 3, 5, 3, 1)^2)$

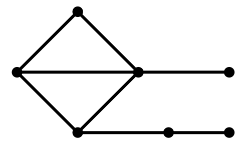
Since a tree T has a unique path joining each pair of nodes, clearly $pds(T) = dds(T)$. In general, $pds(G)$ distinguishes between non isomorphic graphs far more frequently than $dds(G)$ does.



The path degree sequence of the nodes of a graph

Definition 5.25. *Let g_i denotes the number of pairs of nodes joined by i geodesics in graph G. Capobianco defined the geodesic distribution gd(G) of G as $(g_1, g_2, g_3, ...)$. We know that geodetic graph has a unique geodesic joining each pair of nodes. Thus, G is geodestic if and only if $gd(G)$ has a single term.*

The first difference we note between $gd(G)$ and other sequences is that the length of the sequence $gd(G)$ is not specified. This length varies with G and can be quite long. Let $m(p)$ denote the maximum length of $gd(G)$ for a graph G on p nodes.



A graph with geodesic distribution (22,5,1)

Theorem 5.26. *Let $T = \{\langle t_i \rangle\}$ be the set of all partitions of the integer $p-2$. The value $m(p)$ is achieved by maximising $\prod t_i$ over T.*

Lemma 5.27. *In $m(p) = \prod t_i$, each factor t_i is at most 4.*

Lemma 5.28. *For $m(p) = \prod 2^b 3^c$, where $2b + 3c = p - 2$ we have $b \leq 2$.*

5.6 OTHER SEQUENCES

Definition 5.29. *Let n_i denote the number of pairs of nodes with i common neighbours in a graph G. The common neighbour distribution nd(G) of a graph on p nodes as $(n_0, n_1, n_2, ..., n_{p-2})$. This sequence was introduced to aid n distinguishing non isomorphic graphs. For trees T, nd(T) is derivable from $dd(t)$.*

Theorem 5.30. *For a tree T, $\binom{nd(T)=((p}{2)-D_2,D_2)}$.*

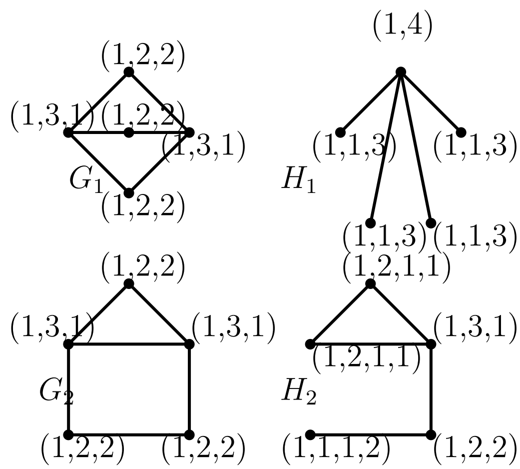
Corollary 5.31. *Let $degv_i$ denote the degree of node v_i in a tree T. Then,*

$$\binom{nd(T)=((p}{2)-\sum_{i=1}^{p}(degv_i 2),(degv_i 2))}.$$

Theorem 5.32. *In any graph G,*

$$\binom{\mu_N(G)=\sum_{i=0}^{p-2}(degv_i}{2)/(p2)}.$$

It was found that when the common neighbour distribution of a graph equals that of its complement, there is a direct relation to dominating sets. Set $X \subset V(G)$ *dominates* set $Y \subset V(G)$ if every node in $Y - X$ is adjacent to a node in X.



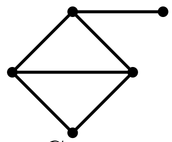
Pairs of graph for whih exactly one of $dd(G)$ and $nd(G)$ agree

Theorem 5.33. *If $nd(G) = nd(\bar{G})$, then n_0 equals the number of dominating sets of order 2 in G.*

Proof : Let $nd(G) = nd(\bar{\mathrm{G}})$. Then for two distinct nodes u, v having no common neighbour, there exists a pair of nodes u', v' with no common neighbours in $\bar{\mathrm{G}}$. Hence $\{u', v'\}$ is a dominating set for G.

∎

Corollary 5.34. *If G is a connected graph with $nd(G) = nd(\bar{G})$ and $n_0 > 0$, then $d(G) \leq 5$.*



$eds(G) = (2, 3, 3, 4, 4, 4)$
$ends(G) = (2, 2, 2, 2, 3, 3)$

G

A graph G with sequences $eds(G)$, $ends(G)$

Theorem 5.35. *For a graph G, $nd(G) = nd(\bar{G})$ if and only if N_i equals the number of apairs of nodes which dominate $n - i$ nodes of G.*

Definition 5.36. *Let $ed(x)$ be the number of edges incident with edge x. Then the edge degree sequence eds(G) of graph G is the sequence $ed_1, ed_2, ..., ed_q$ of values $ed(x)$ arranged in non decreasing order. Note that if $x = uv$, then $ed(x) = degu + degv - 2$. Clearly, $eds(G)$ is precisely the degree sequence of $L(G)$, the line graph of G..*
For each edge $x = uv$, let the edge-to-node degree end(x) of x be the number of distinct nodes in $G - x$ adjacent to either u or v. Then the edge-to-node degree end(G) of G is the sequence of values $end(x)$ listed in non decreasing order.

For each edge $x = uv$, let the edge-to-node degree end(x) of x be the number of distinct nodes in $G - x$ adjacent to either u or v. Then the edge-to-node degree sequence ends(G) of G is the sequence of values $end(x)$ listed in non decreasing order . A graph G along with $eds(G)$ and $ends(G)$ are displayed above.

5.6.1 FURTHER RESULTS

1. The mean distance of $K_{a,b}$ is

$$\mu_D(K_{a,b}) = \frac{2(a^2 - a + b^2 - b + ab)}{(a^2 - a + b^2 - b + 2ab)}$$

2. For nontrivial paths and for cycles, we have

$\mu_D(P_p) = r(P_p)$ if and only if $p = 2$ or $5 (p \geq 2)$.

$\mu_D(C_p) = r(C_p)$ if and only if $p = 3$.

3. For complete bipartite graphs and wheels, we have

$\mu_D(K_{a,b}) = r(K_{a,b})$ if and only if $a = b = 1$.

$\mu_D(W_{1,n}) = r(W_{1,n})$ if and only if $n = 4$.

Definition 5.37. *The independence number of a graph G is the maximum number of nodes in G, no two of which are adjacent. The mean distance of a connected graph is less than or equal to the independence number of G.*

A graph G is bigcodetic if each pair of nodes are joined by at most two geodesics.

Chapter 6

DIGRAPHS

6.1 DIGRAPHS AND CONNECTEDNESS

Definition 6.1. *A digraph D consists of a finite set V of nodes and a collection of ordered pairs of distinct nodes from V. Any such pair (u, v) is called an arc or directed edge and will be denoted by uv. The arc uv goes from u to v and is incident with u and v. We also say that u is adjacent to u and v is adjacent from u. The indegree id(v) of a node v is the number of nodes adjacent to v, and to the outdegree od(v) is the number adjacent from v.*

A directed walk in a digraph D is an alternating sequence of nodes and arcs $v_0, x_1, x_2, ..., x_n, v_n$ in which each arc is $v_{i-1}v_i$. The length of such a walk is n, the number of arcs in it. A closed walk has the same first and last nodes, and a spanning walk contains all the nodes of D. A path is a walk in which all nodes are distinct, a cycle is a non-trivial closed walk with all nodes distinct(except first and last). An acyclic digraph contains no directed cycles. If there is a path from u to v, then v is said to be reachable from u, and the distance $d(u, v)$ from u to

v is the length of any shortest such path.

Definition 6.2. *Each walk is directed from the first node v_o to the last v_n. We also need a concept which does not have this property of direction and is analogous to a walk in a graph. A semiwalk is again an alternating sequence $v_0, x_1, v_1, ..., x_n, v_n$ of nodes and arcs, but each arc x_i may be either $v_{i-1}v_i$ or v_iv_{i-1}. A semipath, semicycle and so forth, are defined as expected.*
Whereas a graph is either connected or is not, there are three different ways in which a digraph may be connected. A digraph is strongly connected or strong, if every two nodes are mutually reachable. it is unilaterally connected or unilateral, if for any two nodes at least one is reachable from the other, and it is weakly connected or weak, if every two nodes are joined by a semipath.Clearly, every strong path is unilateral and every unilateral digraph is weak, but the converse statements are not true. A digraph is disconnected if it is not even weak.

Theorem 6.3. *A digraph is strong if and only if it has a closed spanning walk, it is unilateral if and only if it has a spanning walk, and it is weak if and only if it has a spanning semiwalk.*

Corresponding to connected components of a graph, there are three different kinda of components of a digraph D. A *strong component of D* is a maximal strong of a subgraph; a *unilateral component* and a *weak component* are defined similarly. It is very easy to verify that every node of a digraph D is in just one weak component and in at least one unilateral component and this is also holds for each arc. Furthermore, each node is in exactly one strong component, and an arc lies in one strong component or none, depending on whether or not it is in some cycle.

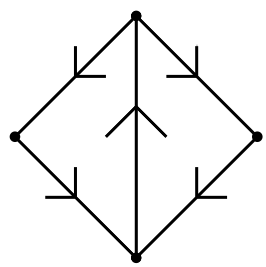
The strong components of a digraph are the most important among these. One reason is the way in which they yield a new digraph which although simpler, retains some of the structural properties of the original. Let $S_1, S_2, ..., S_n$ be the strong components of D. The *condensation* $D*$ of D has the strong components of D as its nodes with an arc from S_i to S_j whenever there is at least one arc in D from a node of S_i to a node of S_j.

Definition 6.4. *A digraph D is symmetric if whenever uv is an arc, then so is vu. On the other hand, D is asymmetric if the presenc of uv is obviates that of vu. When both uv and vu are in D, they form a symmetric pair. This D is symmetric if and only if it has no symmetric pairs. The following figure illustrates digraphs with these properties. The digraph of a graph $G = (V, E)$, written $D(G)$, also has V as its set of node set, and each edge e of G is replaced by the symmetric pair of arcs joining the two endnodes of e. The graph of digraph D, written $G(D)$, also has the same node set as D, but two nodes u and v are now adjacent if they are joined in D it is the symmetric closure of D.*

An orientation of a graph G is any digraph that results from an assignment of directions to the edges of G. If G has at least one edge, any orientation of G is asymmetric and is called oriented graph.

Theorem 6.5. *A graph G has a orientation that is strong if and only if G is connected and has no bridges.*

Since every non-trivial graph G has at least two nodes that are not cutnodes, it follows that every non-trivial digraph does as well. This means that if D is strong, unilateral, or weak then there are two nodes u and V such that both $D - u$ and $D - v$ are weak.



A strong graph with no spanning cycle

It should be stressed that a strong digraph need not have a spanning cycle. For example, the digraph on 4 nodes in the above figure is strong yet has no spanning cycle. However, there is a strong relationship between cycles and strong digraphs as illustrated following

Theorem 6.6. *A weak digraph D is strong if and only if every arc of D is contained in a cycle.*

Proof : If D is strong then for every arc uv there must be a path $v, e_0, v_1, e_1, ..., e_n, u$ from v to u. Then $v, e_0, v_1, e_1, ..., e_n.u, uv, v$ is a cycle containing arc uv. Conversely, it is given that every arc of the weak digraph D is contained in a cycle. Since D is weak, there is a semiwalk $u, e_1, v_1, e_2, ..., e_n, v$ joining any two nodes u and v. arc e_1(which should be either uv_1 or v_1u) is contained in a cycle, so u and v_1 are in the same strong component of D. Similarly each v_i and v_{i+1} are in the same strong component as are v_n and v. Thus all the nodes of the semiwalk joining u and V are in the same strong component of D. As u and v are any two nodes, it follows that D is strong.

∎

6.2 ACYCLIC DIGRAPHS

The *converse digraph* D' of D has the same set of nodes as D and the arc uv is in D' if and only if arc uv is in D. Thus the converse of D is obtained by reversing the direction of every arc of D. We have already encountered some converse concepts, such as indegree and outdegree and these concepts concerned with direction are related by a rather powerful principle. This is a classical result in the theory of binary relations.

Principle of Directional Duality: For each theorem about digraphs, there is a corresponding theorem obtained by replacing every concept by its converse.

Theorem 6.7. *An acyclic digraph has at least one node of outdegree zero.*

Proof : Consider the last node of any longest path in the digraph. This node can have no nodes adjacent from it since otherwise there would be a cycle.
The dual theorem follows immediately by applying the principle of Directional

Duality. In keeping with the use of D' to denote the converse of digraph D, we shall use primes to denote dual results.

∎

Theorem 6.8. *An acyclic digraph D has at least one node of indegree zero.*

It was noted that the condensation of any digraph is acyclic. The adjacency matrix $A(D)$ of a digraph D is a $(0,1)$-matrix with $a_{ij} = 1$ if there is an arc from v_i to v_j.

Theorem 6.9. *The following properties of a digraph D are equivalent.*

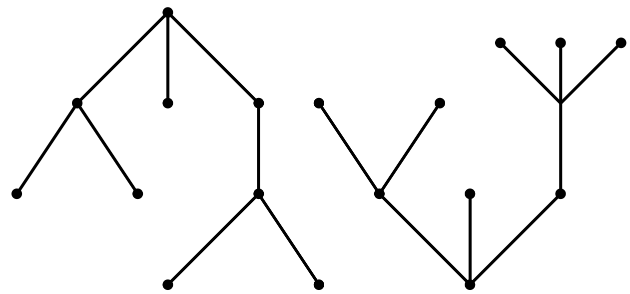
1. D is acyclic.

2. $d*$ is isomorphic to D.

3. Every directed walk of D is a directed path.

4. It is possible to label the nodes of D so that the adjacency matrix $A(D)$ is upper triangular.

Definition 6.10. *Two dual types of acyclic digraphs are of particular interest. A source in D is a node which can reach all others, a sink is the dual concept. An out tree is a digraph with a source but having no semicycles; an in-tree is its dual. The source of an out-tree is its root as is the sink of an in-tree. An out-tree has also been called an arborescence. These concepts have been widely used in computer science in searching and sorting algorithms.*



An out tree and its converse

Theorem 6.11. *A weak digraph is an out-tree if and only if it has exactly one root and all other nodes have indegree one.*

Proof : Suppose that D is a weak digraph. If D is an out-tree, it has exactly one root and no semicycles. Hence, each node is reachable from the root in only one way, so each nonroot has indegree one. On the other hand, if D has exactly one root r and all other nodes have indegree one, then there is a unique directed path from r to each other node, and D has no semicycles. Thus D is an out-tree.

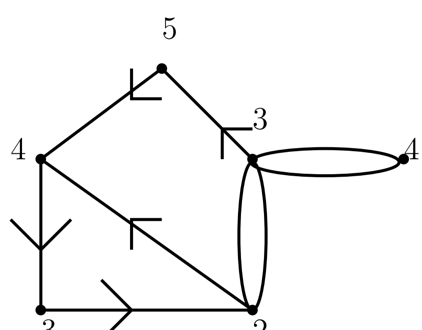
∎

Theorem 6.12. *A weak digraph is an in-tree if and only if it has exactly one root and all other nodes have out-degree 1.*

6.3 LONG PATHS IN DIGRAPHS

The values of certain invariants might guarantee the existence of a path of a given length in a graph. A simpler statement is true for digraphs. For example, every orientation of an n chromatic graph contains a directed path of length $n-1$. We saw that many of the sufficient conditions for hamiltonicity have counterparts when considering generalisations of hamiltonian graphs. We shall now examine these concepts in the concepts of digraphs.

Definition 6.13. *For digraphs, distance concepts are defined analogous to those for graphs except that we must heed the directions on the arcs. Thus, the distance from u to v is the length of a shortest $u-v$ path. The eccentricity of a node v in a digraph D is its distance to afarthest node in D. For a strong digraph the eccentricities are all finite. The radius is the minimum eccentricity and the diameter is the maximum. It is easy to read off the eccentricity of node $v_i \in D$ from the distance matrix $\partial(D)$. The diameter and radius can be obtained similarly.*



A digraph and its eccentricities

Definition 6.14. *A circulant is a graph determined by its order p and a subset $S = \{a, b, c, ...\}$ of $\{1, 2, 3, ..., \langle p/2 \rangle\}$ as follows. The circulant graph $C(p : S) = C(p : a, b, c, ...)$ has node set $Z_p = \{0, 1, ..., p-1\}$ and each node u is adjacent with the nodes $u + a, u + b, u + c, ...,$ all sums taken modulo p. Now the corresponding directed circulant is defined similarly, except that adjacent with is replaced by adjacent to. Certain directed circulants have also been called double loop computer networks namely, those of form $C(p; 1, h)$ which is denoted more briefly by $D(p, h)$. The exact value of the diameter of the directed circulant $D(p, h)$. One wants to minimise the diameter as it varies directly with the transmission time when the nodes are microprocessors and the arcs are communication channels. The choice of jump size h is crucial in such problems.*

Definition 6.15. *A digraph is hamiltonian if it has a closed spanning path. The sufficient conditions for a digraph to be hamiltonian are similar in flavour to those for graphs. Of course, every hamiltonian digraph is strong, but the converse is not true. For example, a strong digraph is strong but the converse is not true. Thus, most results on hamiltonian digraphs begin with the assumption that D is strong. For a node v in a digraph D, let $degv = id(v) + od(v)$.*

Theorem 6.16. *If D is non-trivial strong digraph of order p such that for every pair of distinct nonadjacent nodes u and v*

$$degu + degv \geq 2p - 1$$

then D is hamiltonian.

Outline of proof Since D is strong it has cycles. Let C be a longest cycle in D and suppose that C does not span. Then for a node v not on C there are paths from C to v and back to C either rejoining C at the same node of departure (a) or a different node (b). If all such paths are restricted to type (a), one gets a contradiction. Thus there is a path of type (b). If the path of type (b) is too long one would get a cycle longer that C, a contradiction. Thus the length of the path must be restricted. but the restriction on this length also conflicts. Thus C must in fact span D, hence D is hamiltonian.

Corollary 6.17. *If D is a strong digraph of order p such that $degv \geq p$ for all nodes v in D, then D is hamiltonian.*

Corollary 6.18. *Let D is a nontrivial graph of order p. If every pair of distinct nodes u and v with u not adjacent to v satisfies*

$$id(v) + od(v) \geq p$$

then D is hamiltonian.

Proof : In order to apply Meyniel theorem, we first show that D is strong. For arbitrary nodes u and v, we must show that v is reachable from U. If u is adjacent to v, we are done, so assume the contrary. Then there is anode w adjacent from

u and adjacent to v. Hence v is reachable from u and D is strong.

Now for any two distinct nonadjacent nodes u and v of D, we have

$$degu + degv = id(u) + od(u) + id(v) + od(v)$$

$$=\text{id(u)+od(v)+id(u)+od(v)} \geq p + p \geq 2p - 1$$

Hence D is hamiltonian.

∎

Corollary 6.19. *If D is a digraph of order p such that for all pairs of nonadjacent nodes u and v*

$$degu + degv \geq 2p - 3$$

then D has a spanning path.

Proof : The very first theorem guarantees that D is at least weak. We can form a strong digraph D_1 as the symmetric join $D + K_1$ of a new node w to D, that is, add w and a symmetric pair of arcs between w and each node of D. For every pair of nodes u_1 and v_1 in D_1 we have

$$degu_1 + degv_1 \geq 2p - 3 + 4 = 2p - 1 = 2(p + 1) - 1$$

As D_1 has order $p + 1$, the above theorem implies that D_1 has a hamiltonian cycle C. By deleting node w and its incident arcs in C, we obtain a spanning path of D.

∎

Definition 6.20. *The first generalisation of hamiltonian digraphs we consider are digraphs for which there is a spanning path from each node to each other node. A digraph D is hamiltonian-connected if there is a spanning $u - v$ path for all pairs of distinct nodes u and v in D. A hamiltonian-connected digraph is always hamiltonian, but the converse is not true as a directed cycle of order at least 4 shows.*

Theorem 6.21. *Let D be a nontrivial digraph of order p. If every pair of distinct nodes u and v with u not adjacent to v satisfies*

$$od(u) + id(v) \geq p + 1$$

then D is hamiltonian-connected.

Definition 6.22. *There is a natural analogy to the concept of a pancyclic graph. A digraph D of order p is pancyclic if D contains a directed cycle of each length k, $3 \leq k \leq p$. Thus, these digraphs are a special class of hamiltonian digraphs.*

Theorem 6.23. *Let D be a strong digraph of order $p \geq 3$ such that $degu + degv \geq 2p$ for all pairs u and v of nonadjacent nodes. then D is either pancyclic or p is even and D is the digraph of $K_{p/2,p/2}$.*

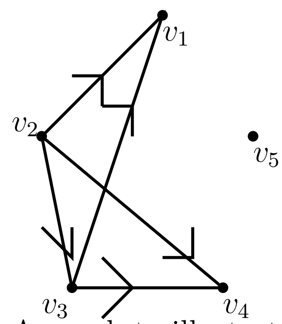
There are also results for strong digraphs which guarantee the existence of a path of a given length when the digraph might not have a spanning path. For example, a strong digraph of order p with $id(v) \geq k$ and $od(v) \geq h$ for all v contains a path of length at least $min\{h+k, p-1\}$.

6.4 MATRICES AND EULERIAN DIGRAPHS

There are several matrices associated with a digraph, and each one provides certain information about the digraph. For example the row sums of the adjacency matrix $A(D)$ of the digraph gives the outdegree of the nodes of D, while the column sums give the indegrees.
As in the case of graphs, the powers of the adjacency matrix A of a digraph give information about the number of walks from one node to another.

Theorem 6.24. *The i, j entry $a_{ij}^{(}n)$ of A^n is the number of walks of length n from v_i to v_j.*

Three other matrices are associated with D are the reachability matrix, the distance matrix, and the detour matrix. In the *reachability matrix* $R(D)$, $r_{ij} = 1$ if v_j is reachable from v_i and 0 otherwise. The i, j entry of the *distance matrix* $\partial(D)$ gives the distance from node v_i to node v_j, and is ∞ if there is no path from v_i to v_j. In the *detour matrix* $T(D)$, the i, j entry is the length of any longest path from v_i to v_j and again is ∞ if there is no such path.



A graph to illustrate three
associated matrices

Corollary 6.25. *The entries of the reachability and distance matrices can be obtained from the powers of A as follows.*

1. For all $i, r_{ii} = 1$ and $d_{ii} = 0$.

2. $r_{ij} = 1$ if and only if for some $n, a^{(}_{ij}n) > 0$.

3. $d(v_i, v_j)$ is the least n such that $a^{(}_{ij}n) > 0$, and 0 otherwise.

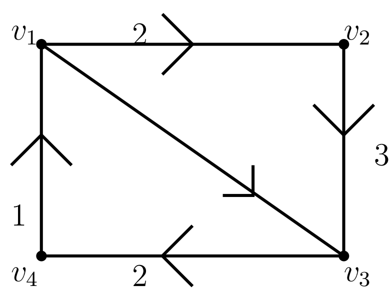
Definition 6.26. *The element wise product sometimes called the Hardamard product $B \times C$ of two matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ has $b_{ij}c_{ij}$ as its i, j entry. The reachability matrix can be used with elementwise products to find strong components.*

Corollary 6.27. *Let v_i be a node of a digraph D. The strong componenet of D containing v_i is determined by the unit entries in the ith row of the symmetric matrix $R \times R^T$.*

As in case of graphs, weights are placed on the arcs corresponding to a directed distance from one node to the another. If for some pair of nods v_i

and v_j there is no intermediate node v_k such that $d_{ij} = d_{ik} + d_{kj}$ then d_{ij} is called a basic distance. Thus each basic distance d_{ij} is determined by the weight on the single arc $v_i v_j$. In the following figure , $d_{13} = 4$ and is basic, but d_{24} is not because $d_2 4 = d_2 3 + d_3 4$.

A weighted graph W is a realisation of matrix M of order n if there is a subset $V = \{v_1, v_2, ..., v_n\}$ of the nodes of W having order n such that $d(v_i, v_j) = m_{ij} \forall i, j, 1 \leq i, j \leq n$. A realisation is optimal if the total weights used on its arcs is a minimum. Optimal realisations of directed distance matrices of order n were characterised.



A weighted digraph

Theorem 6.28. *A directed distance matrix M of order n has an optimal realisation if and only if M can be realised by a simple directed cycle, or equivalently, M has n basic entries.*

Definition 6.29. *The number of spanning in-trees to a given node ina digraph was found. To give this result, called the matrix tree theorem for digraphs, we need some other matrices related to D. Let M_{od} denote the matrix obtained from $-A$ by replacing with the ith diagonal entry by $od(v_i)$. The matrix M_{id} is defined* *dually.*

Theorem 6.30. *For any labelled digraph D , the value of the cofactor of each entry in the ith orw of M_{od} is the number of spanning in-trees with v_i, as sink.*

Theorem 6.31. *The value of the cofactor of any entry in the jth column of M_{id} is the number of spanning out trees with root v_i.*

Definition 6.32. *An eulerian trail in a digraph D is a closed spanning walk in which each arc of D occurs exactly only once. A digraph is eulerian if it has such a walk.*

Theorem 6.33. *For any weak digraph D, the following statements are equivalent.*

1. D is eulerian.

2. For each node $u, od(u) = id(u)$.

3. There exists a partition of the arc set of D into directed cycles.

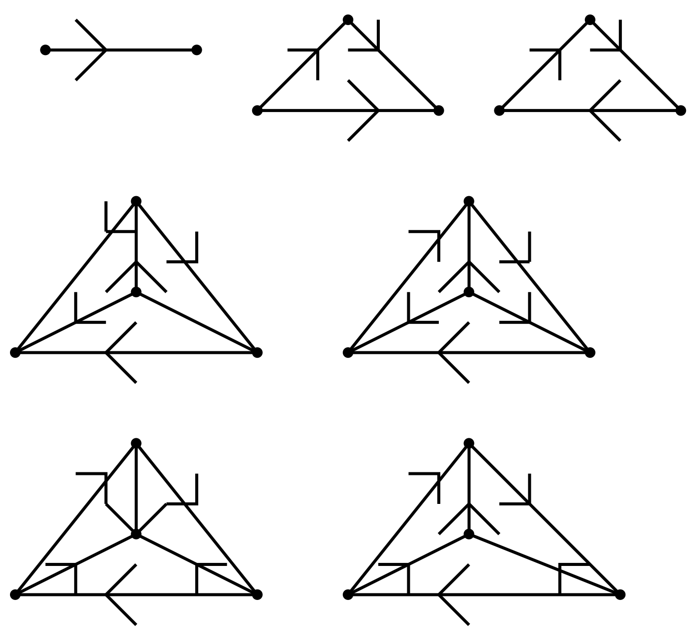
Corollary 6.34. *In an eulerian graph D, let $d_i = id(v_i)$ and c be the common value of all the cofactors of M_{od}. Then the number of eulerian trails is*

$$c. \prod_{i=1}^{p} (d_i - 1)!$$

6.5 TOURNAMENTS

Perhaps the most studied digraphs are the tournaments. A *tournaments* is a non-trivial oriented complete graph. All tournaments with two, three and four nodes are shown in the following figure. The first with three nodes is called *transitive triple*, the second a *cyclic triple*.

In a round-robin tournament, a given collection of players or teams play a game in which the rules of the game do not allow for a draw. Every pair of players encounter each other and exactly one from each pair emerges victorious. The players represented by nodes and for each pair of nodes an arc its drawn from the winner to loser, resulting in a tournament.



The seven smallest tournaments

Theorem 6.35. *Every tournament has a spanning path.*

Proof : The proof is by induction on the number of nodes. Every tournament with 2,3 or 4 nodes has a spanning path, by inspection. Assume the result is true for all tournaments with n nodes and consider a tournament T with $n+1$ nodes. Let v_0 be any node of T. Then $T - v_0$ is a tournament with n nodes, so it is a spanning path P, say $v_1v_2...v_n$. Either arc v_1v_0 or $v_1v_0 \in T$. If $v_0v_1 \in T$, then $v_0v_1...v_n$ is a spanning path of T. If $v_1v_0 \in T$, then let v_k be the first node of P for which the arc v_0v_k is in T. If no such node v_k exists, then $v_1v_2...v_nv_0$ is a spanning path. In any case, we have shown that T has spanning path completing the proof.

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Definition 6.36. *Using the terminology from robin-robin tournaments, we say that the score of a node in a tournament is its outdegree ans a node is said to dominate each node to which it is adjacent.*

Theorem 6.37. *In any tournament the distance from a node with maximum score to any other node is 1 or 2.*

Proof : Let v be a node with maximum score k in a tournament T. Suppose u is a node at distance at least 3 from v. Then $uv \in T$ and u must dominate each of the k nodes that v dominates. Hence $od(u) \geq k+1 > od(v)$, a contradiction.

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Theorem 6.38. *A non decreasing sequence of non-negative integers $s_1, s_2, ..., s_p$ is the score sequence of a tournament T if and only if for each k, $1 \leq k \leq p$, we* *have*

$$\binom{\sum_{i=1}^{k} s_i \geq k}{2}$$

with equality holding for $k = p$.