WELCOME
Continuity and Uniform continuity using Epsilon – Delta Property

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Outline

1. Motivation
2. Sequence
3. Convergence
4. Continuous function
5. Uniform Continuous
What is set?.
Introduction to Sets

What is a set?.

Is it merely a collection of objects or "things"?.
Introduction to Sets

For Example
The items you wear: shoes, socks, hat, shirt, pants, and so on.
Introduction to Sets

For Example
The items you wear: shoes, socks, hat, shirt, pants, and so on. I’m sure you could come up with at least a hundred.
For Example

The items you wear: shoes, socks, hat, shirt, pants, and so on. I’m sure you could come up with at least a hundred. This is known as a set.
Introduction to Sets

For Example

The items you wear: shoes, socks, hat, shirt, pants, and so on. I’m sure you could come up with at least a hundred. This is known as a set.
Introduction to Sets

For Example
Types of fingers.
For Example

Types of fingers. This set includes index, middle, ring, and pinky.
Introduction to Sets

For Example

Types of fingers. This set includes *index*, *middle*, *ring*, and *pinky*.
Introduction to Sets

For Example

Types of fingers. This set includes index, middle, ring, and pinky.

So it is just things grouped together with a certain property in common.
Introduction to Sets

What is set?
Well, simply put, it’s a collection.
Introduction to Sets

What is set?
Well, simply put, it’s a collection.

Definition
A set is a collection of well defined objects or things.
Introduction to Sets

Notations

Sets are generally denoted by capital letters $A, B, C, \cdots$ etc.,
Introduction to Sets

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Elements of the sets are denoted by the small letters $a, b, c, d, e, f, \ldots$ etc.,
Introduction to Sets

Notations

- Sets are generally denoted by capital letters $A, B, C, \cdots$ etc.,
- Elements of the sets are denoted by the small letters $a, b, c, d, e, f, \cdots$ etc.,
- If $x$ is an element of the set $S$, then it is written as $x \in S$ and read as $x$ belongs to $S$. 
Introduction to Sets

Notations

- Sets are generally denoted by capital letters $A, B, C, \cdots$ etc.,
- Elements of the sets are denoted by the small letters $a, b, c, d, e, f, \cdots$ etc.,
- Is $x$ is an element of the set $S$, then it is written as $x \in S$ and read as $x$ belongs to $S$.
- If $x$ is a not the member of the set $S$, then it is written as $x \notin S$ and read as $x$ does not belong to $S$. 
Introduction to Sets

Example
Consider the set $V = \{a, e, i, o, u\}$

$a \in V$, $i \in V$ but $b \notin V$
Example

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$V$ is the set of vowels in alphabet.
Example

Consider the set \( V = \{a, e, i, o, u\} \)
\( a \in V \), \( i \in V \) but \( b \notin V \)

\( V \) is the set of vowels in alphabet.

Is it?

Girls are brilliant.
**Introduction to Sets**

**Example**

Consider the set \( V = \{a, e, i, o, u\} \)

\( a \in V, \ i \in V \) but \( b \notin V \)

\( V \) is the set of vowels in alphabet.

Is it ?

Girls are brilliant.

Is it a set ?
Motivation

Introduction to Sets

Example
Consider the set $V = \{a, e, i, o, u\}$
$a \in V$, $i \in V$ but $b \notin V$
$V$ is the set of vowels in alphabet.

Is it?
Girls are brilliant.
Is it a set?
No, because here brilliant is not defined.
Sets

\[\mathbb{N} \quad - \quad \text{Natural Numbers} \quad \{1, 2, 3, 4, \ldots\}\]
Sets

\( \mathbb{N} \) - Natural Numbers \( \{1, 2, 3, 4, \ldots \} \)

\( \mathbb{Z} \) - Set of Integers \( \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots \} \)
Motivation

Sets

\[ \mathbb{N} \] - Natural Numbers \{1, 2, 3, 4, \ldots \}

\[ \mathbb{Z} \] - Set of Integers \{0, \pm1, \pm2, \pm3, \pm4, \cdots \}

\[ \mathbb{Q} \] - Set of Rational Numbers \{\frac{p}{q}, q \neq 0\}
Motivation

Sets

\( \mathbb{N} \) - Natural Numbers \( \{1, 2, 3, 4, \cdots \} \)
\( \mathbb{Z} \) - Set of Integers \( \{0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots \} \)
\( \mathbb{Q} \) - Set of Rational Numbers \( \left\{ \frac{p}{q}, q \neq 0 \right\} \)
\( \mathbb{R} \) - Set of Real Numbers \( (-\infty, \infty) \)
Motivation

Graphical View

\[ \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \]

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Graphical View

\[ \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \]
Motivation

Graphical View

$N \subset Z \subset \mathbb{Q} \subset \mathbb{R}$
Graphical View

\[ \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \]
Motivation

Graphical View

\[ N \subset Z \subset Q \]

\[ Q \subset R \]

\[ N \subset Z \subset Q \]
Graphical View

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
Graphical View

\[ \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \]
Function

- Function - Relation between two non-empty sets.
Function

- Function - Relation between two non-empty sets.
- Let $A$ and $B$ be two non-empty sets. A function or mapping $f$ from $A$ into $B$ is a rule which assigns each element $a \in A$ a unique element $b \in B$. 
Function

- Function - Relation between two non-empty sets.
- Let $A$ and $B$ be two non-empty sets. A function or mapping $f$ from $A$ into $B$ is a rule which assigns each element $a \in A$ a unique element $b \in B$.
- In mathematically written as $f : A \rightarrow B$ defined by $f(a) = b$ for all $a \in A$. 
Consider the function $f : A \rightarrow B$ by $f(a) = b$
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Consider the function $f : A \rightarrow B$ by $f(a) = b$
Consider the function $f : A \to B$ by $f(a) = b$.

A is called the domain of $f$. 

\[ A \xrightarrow{f} B \] 

\[ a \xrightarrow{f(a)} b \]
Consider the function $f : A \rightarrow B$ by $f(a) = b$.

- $A$ is called the domain of $f$.
- $B$ is called the co-domain of $f$. 
Consider the function $f : A \to B$ by $f(a) = b$

- $A$ is called the domain of $f$
- $B$ is called the co-domain of $f$
- The element $b \in B$ is called the image of $a$ under $f$. 
Consider the function \( f : A \to B \) by \( f(a) = b \)

- \( A \) is called the domain of \( f \)
- \( B \) is called the co-domain of \( f \)
- The element \( b \in B \) is called the image of \( a \) under \( f \).
- The element \( a \in A \) is called the pre-image of \( b \) under \( f \).
Graphical
Graphical

Is it function?

Yes
Graphical
Graphical

Motivation

Is it function? Yes

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Graphical

Motivation

Is it function?

Yes

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Graphical

Motivation

Is it function? Yes

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Graphical
Graphical
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Is it function? Yes

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Graphical

Is it function?
Motivation

Graphical

Is it function?
Yes
Graphical
Graphical
Graphical
Graphical

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) [circle,fill=red,inner sep=1pt] {1};
  \node (B) at (0,-1) [circle,fill=red,inner sep=1pt] {2};
  \node (C) at (0,-2) [circle,fill=red,inner sep=1pt] {3};
  \node (D) at (0,-3) [circle,fill=red,inner sep=1pt] {4};
  \draw (A) -- (B) -- (C) -- (D);

  \node (E) at (1,0) [circle,fill=red,inner sep=1pt] {1};
  \node (F) at (1,-1) [circle,fill=red,inner sep=1pt] {3};
  \node (G) at (1,-2) [circle,fill=red,inner sep=1pt] {5};
  \draw (E) -- (F) -- (G);
\end{tikzpicture}
\end{center}
Graphical

\[ A \xrightarrow{f} B \]

1, 2, 3, 4 \rightarrow 1, 3, 5

Is it function? No
Graphical

Is it function?
No
Graphical

Is it function?

No

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Graphical

\[ A \xrightarrow{f} B \]

- Graphical representation of a function.
- Set \( A \) and set \( B \) with elements 1, 2, 3, 4, 5.
- Function \( f \) maps elements from \( A \) to \( B \).

Is it function?
No
Graphical

Is it function?
Graphical

Is it function?  
No
Graphical
Graphical

A

B

Is it function? If it is, what type is it?

One-to-one (or) Injective

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Graphical

Is it a function? If it is, what type is it?

One-to-one (or) Injective
Graphical

Is it a function? If it is, what type is it?

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\[ A \xrightarrow{f} B \]

1. 1
2. 2
3. 3

4. 2
5. 4
6. 6
7. 8
Motivation

Graphical

Is it a function? If it is, what type is it?

One-to-one (or) Injective

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Is it a function? If it is, what type is it?

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Graphical

Is it a function? If it is, what type is it?

One-to-one (or) Injective
Is it a function? If it is, what type is it?
Graphical

Is it function? If it is, what type is it?

One-to-one (or) Injective
Graphical
Graphical

Is it a function? If yes, what type is it?

Onto (or) Surjective
Motivation

Graphical

Is it function? If it is, what type is it?

Onto (or) Surjective

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Is it a function? If it is, what type is it?

Onto (or) Surjective
Graphical

Is it a function? If it is, what type is it?

Onto (or) Surjective

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Is it a function? If so, what type is it?

Onto (or) Surjective

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Motivation

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Is it a function? If it is, what type is it?

Onto (or) Surjective
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Is it function? If it is, what type is it?

Onto (or) Surjective

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Is it function? If it is, what type is it?
Graphical

Is it function? If it is, what type is it?

Onto (or) Surjective
Graphical

Constant Function $f: \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 3$ is called a constant function. The range of $f$ is 3.
Motivation

Graphical

\[ R \rightarrow R \]

Constant Function \( f: \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(x) = 3 \) is called a constant function. The range of \( f \) is 3.
Motivation

Graphical

$\mathbb{R}^1 \rightarrow \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$

Constant Function $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(x) = 3$ is called a constant function. The range of $f$ is 3.

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Motivation

Graphical

$\mathbb{R}$

1.

2.

$n-1$.

$n$.

$\mathbb{R}$

1.

2.

3.

4.

5.
Graphical

Motivation

Constant Function

$f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3$ is called a constant function. The range of $f$ is 3.
Graphical

The diagram illustrates a function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3$ for $x \in \{1, 2, \ldots, n\}$. The range of $f$ is $\{1, 2, 3, 4, 5\}$.
Graphical

A constant function is defined as $f: \mathbb{N} \to \mathbb{N}$ with $f(x) = 3$. The range of $f$ is 3.
Motivation

Graphical

\[ \mathbb{R} \rightarrow \mathbb{R} \]

1 2 3 4 5

\[ f(x) = 3 \]

Constant Function

The range of \( f \) is 3.
Graphical

A constant function $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(x) = 3$ is called a constant function. The range of $f$ is 3.
Constant Function \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(x) = 3 \) is called a constant function. The range of \( f \) is 3.
Graphical

Constant Function

$f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 3$ is called a constant function. The range of $f$ is 3.
Introducing Sequence

In maths, we call a list of numbers in order a sequence.
Introducing Sequence

- In maths, we call a list of numbers in order a sequence.
- Each number in a sequence is called a term.
Introducing Sequence

In maths, we call a list of numbers in order a sequence.

Each number in a sequence is called a term.

If terms are next to each other they are referred to as consecutive terms.
Introducing Sequence

- In maths, we call a list of numbers in order a sequence.
- Each number in a sequence is called a term.
- If terms are next to each other they are referred to as consecutive terms.
- When we write out sequences, consecutive terms are usually separated by commas.
Example
Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]
Example

Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \]

Graphical
Example
Consider the following collection of real numbers given by

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Graphical
Example

Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]

Graphical

\[ \begin{array}{cccccc}
01 & 1 & 1 & 1 & 1 & 1 \\
\hline
n & 5 & 4 & 3 & 2 & 1 \\
\end{array} \]
Example

Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]

Graphical
Example
Consider the following collection of real numbers given by

\[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\]

Graphical

\[\begin{array}{cccccc}
0 & 1 & 1 & 1 & 1 & 1 \\
\hline
n & 5 & 4 & 3 & 2 & 1 \\
\end{array}\]
Example

Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]

Graphical

\[ n^{th} \text{ Term} \]

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 1 & 1 \\
\hline
n & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\]
Example

Consider the following collection of real numbers given by

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \]

This is an example of sequence of real numbers.
Sequence is a function whose domain is the set of natural numbers.
Sequence is a function whose domain is the set of natural numbers.

**Definition**

Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a function and $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots, a_n, \ldots$, is called the sequence in $\mathbb{R}$ determined by the function $f$ and is denoted by $\{a_n\}$, $a_n$ is called the $n^{th}$ term of the sequence.
Convergence of a Sequence

We say that a sequence \((x_n)\) converges if there exists \(x_0 \in \mathbb{R}\) such that for every \(\epsilon > 0\), there exists a positive integer \(N\) (depending on \(\epsilon\)) such that \(x_n \in (x_0 - \epsilon, x_0 + \epsilon)\) for all \(n \geq N\).
Definition

Let \( \{a_n\} \) be a sequence of real numbers.
Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \to l \)
Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \to l \) iff given \( \epsilon > 0 \)
Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \rightarrow l \) iff given \( \epsilon > 0 \)
Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \to l \) iff given \( \epsilon > 0 \) there exists a natural number \( N \) such that

\[
|a_n - l| < \epsilon
\]

for all \( n \geq N \).
Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \rightarrow l \) iff given \( \epsilon > 0 \) there exists a natural number \( N \) such that \( a_n \in (l - \epsilon, l + \epsilon) \) for all \( n \geq N \).
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Definition

Let \( \{a_n\} \) be a sequence of real numbers. \( \{a_n\} \rightarrow l \) iff given \( \epsilon > 0 \) there exists a natural number \( N \) such that \( a_n \in (l - \epsilon, l + \epsilon) \) for all \( n \geq N \).
Convergence of a sequence $a_n$


\[ a_n \]

\[ L \pm \epsilon \]

\[ N \pm \epsilon \]
Convergence

\[ a_n \]

\[ L \pm \epsilon \]

\[ N \]

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Convergence

\[ L \pm \epsilon \]

\[ N \pm \epsilon \]

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Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L \]

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Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L \]

\[ L - \epsilon \]
Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L \]

\[ L - \epsilon \]

\[ N \]
Convergence

\[ L - \epsilon \leq a_n \leq L + \epsilon \]

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Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L \]

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$a_n$

$L + \epsilon$

$L$

$L - \epsilon$

0 2 4 6 8 10 12 14 16 18 20
Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L \]

\[ L - \epsilon \]

\[ N \]

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Convergence

\[ a_n \]

\[ L + \epsilon \]

\[ L - \epsilon \]

\[ N \]

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For any $\epsilon > 0$, there exists a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$.
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For any $\varepsilon > 0$, there exists a positive integer $N$ such that $|a_n - L| \leq \varepsilon$ for all $n > m$. 

Graphical View

$\mathbb{R}$

$L + \varepsilon$

$L - \varepsilon$

$N + \varepsilon$

$N - \varepsilon$

$a_n$
For any $\epsilon > 0$, there exists a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$. 

**Continuity and Uniform continuity**

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For any $\epsilon > 0$, there exists a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$.
For any \( \epsilon > 0 \), there exists a positive integer \( N \) such that \( \left| a_n - L \right| \leq \epsilon \) for all \( n > m \).
For any $\epsilon > 0$, 

\[ |a_n - L| \leq \epsilon \] 

for all $n > N$. 

**Graphical View** 

- $a_n$ 
- $L + \epsilon$ 
- $L$ 
- $0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$
For any $\epsilon > 0$,
Graphical View

For any $\epsilon > 0$, $\exists$ a positive integer $N$
For any $\epsilon > 0$, $\exists$ a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$. 
For any $\epsilon > 0$, there exists a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$. 
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For any $\epsilon > 0$, $\exists$ a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$. 

Graphical View
For any $\epsilon > 0$, $\exists$ a positive integer $N$ such that $|a_n - L| \leq \epsilon$ for all $n > m$. 
Properties of sequence

1. A sequence cannot converge to two different limits.
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2. A sequence converges to real number $A$ and $B$ then $A = B$. 
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Properties of sequence

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3. Any convergent sequence is a bounded sequence. Converse is not true. Example: $\{(-1)^n\}$ is a bounded sequence but not a convergent sequence.
Properties of sequence

1. A sequence cannot converge to two different limits.
2. A sequence converges to real number $A$ and $B$ then $A = B$.
3. Any convergent sequence is a bounded sequence. Converse is not true. Example: $\{(−1)^n\}$ is a bounded sequence but not a convergent sequence.
4. Any convergent sequence is bounded.
Concept
Continuous functions are functions that take nearby values at nearby points.
Origin

The term continuous has been used since the time of Newton to refer to the motion of bodies or to describe an unbroken curve.
Origin

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- Work of Bernard Bolzano in 1817 and Cauchy 1821 identified continuity as a very significant property of function.
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Work of Bernard Bolzano in 1817 and Cauchy 1821 identified continuity as a very significant property of function.

The concept is tied to that of limit, it was the careful work of Weierstrass in the 1870s that brought proper understanding to the idea of continuity.
Continuous function

Let $f : A \rightarrow R$, where $A \subset R$, and suppose that $c \in A$. Then $f$ is continuous at $c$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies that $|f(x) - f(c)| < \varepsilon$. 
Graph
Note
A function \( f : A \rightarrow R \) is continuous on a set \( B \subset A \) if it is continuous at every point in \( B \), and continuous if it is continuous at every point of its domain.
Steps

1. Take \( |f(x) - f(c)| < \varepsilon \) and rewrite it to match \( |x - c| < \delta \) to create a direct relationship
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2. Let $|x - c| < \delta$ and prove $|f(x) - f(c)| < \varepsilon$
Continuous function

The function $\sin x : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$. 
Sinx curve
Continuous function

Choose $\delta = \varepsilon$ in the definition of continuity for every $c \in \mathbb{R}$
Continuous function

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sin(1/x)$, if $x \neq 0$, $f(x) = 0$, if $x = 0$ is continuous on $\mathbb{R} - 0$, since it is the composition of $x \mapsto 1/x$, which is continuous on $\mathbb{R} - 0$ and $y \mapsto \sin y$, which is continuous on $\mathbb{R}$. 
$Sin\left(\frac{1}{x}\right)$ curve
Continuous function

The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
f(x) = x \sin(1/x), \text{ if } x \neq 0, \quad f(x) = 0, \text{ if } x = 0.
\]
Then \( f \) is continuous at 0.
\( x \sin\left(\frac{1}{x}\right) \) curve

\[
f(x) = \begin{cases} 
  x \sin\left(\frac{1}{x}\right), & x \neq 0; \\
  0, & x = 0. 
\end{cases}
\]
Continuous function

The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 
  x \sin(1/x), & \text{if } x \neq 0, \\
  0, & \text{if } x = 0.
\end{cases}
\]
Then \( f \) is continuous on \( \mathbb{R} \setminus \{0\} \).
Continuous function

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

\[ f(x) = x^2 \sin\left(\frac{1}{x}\right), \text{ if } x \neq 0, \quad f(x) = 0, \text{ if } x = 0. \]

Then $f$ is continuous at $0$. 

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Continuity and Uniform continuity

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$x^2 \sin\left(\frac{1}{x}\right)$ curve
Continuous function

The function $f : [0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$. (i) Prove that $f$ is continuous at $c > 0$, we can choose $\delta = \sqrt{c}\varepsilon > 0$

(ii) Prove that $f$ is continuous at 0, we note that if $0 \leq x < \delta$ where $\delta = \varepsilon^2 > 0$, 
\[ f(x) = \sqrt{x} \text{ curve} \]
Continuous function

The function $f(x) = x^2 + 1$ is continuous at $x = 2$
$x^2$ curve
Uniform Continuous function

Let $f : A \rightarrow R$, where $A \subset R$. Then $f$ is uniformly continuous on $A$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ and $x, y \in A$ implies that $|f(x) - f(y)| < \varepsilon$. 
Remarks

❄ The key point of this definition is that $\delta$ depends only on $\varepsilon$, not on $x, y$. 
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❄ A uniformly continuous function on $A$ is continuous at every point of $A$, but the converse is not true.
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* The key point of this definition is that $\delta$ depends only on $\varepsilon$, not on $x, y$.
* A uniformly continuous function on $A$ is continuous at every point of $A$, but the converse is not true.
Continuous function

The sine function is uniformly continuous on \( R \), since we can take \( \delta = \varepsilon \) for every \( x, y \in R \).
Continuous function

Define \( f : [0, 1] \longrightarrow \mathbb{R} \) by \( f(x) = x^2 \). Then \( f \) is uniformly continuous on \([0, 1]\).
$x^2$ curve
Continuous function but not uniform

The function $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbb{R}$.
Continuous function

The function $f : (0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, 1]$. 
$f(x) = \frac{1}{x}$ curve
Continuous function but not uniform

Define $f : (0, 1] \rightarrow \mathbb{R}$ by $f(x) = \sin\left(\frac{1}{x}\right)$

Then $f$ is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$. 
Time to Interact
Thank You