RELATIONS AND FUNCTIONS

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Topics

- Relation
- Properties of relation in a set
- Relation Matrix and the Graph of a relation
- Equivalence relation
- Compatibility relation
- Composition of binary relations
- Functions
- Types of functions

Introduction

- Relationship between **elements of sets** is represented using a mathematical structure called relation.
- The most intuitive way to describe the relationship is to represent in the form of ordered pair.

Definition :

Let A and B be two sets. A binary relation from A to B is a subset of A x B.

- Note: If A, B and C are three sets, then a subset of AxBxC is known as ternary relation. Continuing this way a subset of $A_1xA_2x\ldots xA_n$ is known as n – ary relation.

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- Let A and B be two sets. Suppose R is a relation from A to B (i.e. R is a subset of A x B). Then, R is a set of ordered pairs where each first element comes from A and each second element from B.
  - Thus, we denote it with an ordered pair (a, b), where $a \in A$ and $b \in B$.
    - ie.,$R=\{(a,b) / a \in A \text{ and } b \in B\}$
- We also denote the relationship with a R b, which is read as "a related to b".

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- Consider the following example :
  - A={Mohan, Charles, David, Ravi}
  - B={Kavitha, Marry, Chithra}
- Suppose Kavitha has two brothers Mohan and Charles, Marry has one brother David, and Chitra has one brother Ravi.
- If we define a relation R " is a brother of" between the elements of A and B then clearly.
  - Mohan R Kavitha, Charles R Kavitha, David R Marry, Ravi R Chitra.
- After omitting R between two names these can be written in the form of ordered pairs as :
  - (Mohan, Kavitha), (Charles, Kavitha), (David, Marry), (Ravi, Chitra).

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- The above information can also be written in the form of a set R of ordered pairs as
  - R= {(Mohan, Kavitha), (Charles, Kavitha), (David, Marry), (Ravi, Chitra)}
  - Clearly R⊆ AxB, i.e.R={(a,b)/a∈A and b∈B }
- **Domain and Range of a Relation**
  - If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.
  - Consider previous example given above.
    - Domain = {Mohan, Charles, David, Ravi}
    - Range = {Kavitha, Marry, Chitra}

Properties of relation in a set

- reflexive
- symmetric
- transitive
- irreflexive
- anti symmetric
- asymmetric
- equivalence relation

Reflexive

Definition:

A binary relation R in a set X is reflexive if x R x, for every x Є X

- That is (x, x) Є R

Example:

If R_1 = {(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)} be a relation on A = {1, 2, 3}, then R_1 is a reflexive relation, since for every x ∈ A, (x, x) ∈ R_1.

Symmetric

Definition:

A relation R in a set X is symmetric if x R y, then y R x for every x and y in X.

Example:

If R_3 = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3, 1)} be a relation on A = {1, 2, 3}, then R_3 is a symmetric relation.

Transitive

Definition

A relation R in a set X is transitive if, for every x, y, and z are in X, whenever x R y and y R z , then x R z. That is $(x,z) \in R$.

Example:

Let A = {a, b, c, d} and R be defined as follows: R = {(a, b), (d, b), (b, d), (a, d), (b, c), (d, c)}. Here R is transitive relation on A.

irreflexive

Definition

A relation R in a set X is irreflexive if, for every x Є X , (x, x)∉X.

Example:

Let A be a set of positive integers and R be a relation on it defined as, a R b if "a is less than b". Then, R is an irreflexive relation, as a is not less than itself for any positive integer a.

Anti symmetric

Definition

A relation R in a set x is anti symmetric if , for every x and y in X, whenever x R y and y R x, then x = y.

Example:

- The relation "less than or equal to (≤)", is an anti-symmetric relation.
- Let A = {1, 2, 3, 4} and R be defined as: R = {(1, 1), (2, 2), (3, 3), (4, 4)}. Here R is antisymmetric relation.

asymmetric relation

Definition:

Let R be a relation defined from a set A to itself. For a, b ∈A, if a R b, then b $\not R$ a , then R is said to be asymmetric relation.

Example:

- Let A = {a, b, c, d} and R be defined as: R = {(a, b), (b, c), (b, d), (c, d), (d, a)}. Here R is asymmetric relation.

equivalence relation

Definition:

Let R be a relation defined from a set A to itself. If R is reflexive, symmetric and transitive, then R is called as equivalence relation.

Example:

Consider the set L of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other. This relation is an equivalence relation.

Relation Matrix and the Graph of a relation

- **Relation Matrix**: A relation R from a finite set X to a finite set Y can be represented by a matrix is called the relation matrix of R.
- Let X = $\{x_1, x_2, \ldots, x_m\}$ and Y = $\{y_1, y_2, \ldots, y_n\}$ be finite sets containing m and n elements, respectively, and R be the relation from A to B. Then R can be represented by an m × n matrix $M_R = [r_{ij}]$, which is defined as follows:

$$r_{ij} = \begin{cases} 1, & \text{if } (x_i, y_j) \in R \\ 0, & \text{if } (x_i, y_j) \notin R \end{cases}$$

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Example:

- Let $A = \{1, 2, 3, 4\}$ and $B = \{b_1, b_2, b_3\}$. Consider the relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$. Determine the matrix of the relation.

Solution:

- $A = \{1, 2, 3, 4\}$, $B = \{b_1, b_2, b_3\}$.
- Relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$. Matrix of the relation R is written as
- That is $M_R =$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

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Graph of a Relation:

- A relation can also be represented pictorially by drawing its graph.
- Let R be a relation in a set $X = \{x_1, x_2, \ldots, x_m\}$. The elements of X are represented by points or circles called nodes.
- These nodes are called vertices. If $(x_i, x_j) \in R$, then we connect the nodes xi and xj by means of an arc and put an arrow on the arc in the direction from xi to x_j . This is called an edge.
- If all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, then we get a graph of the relation R.

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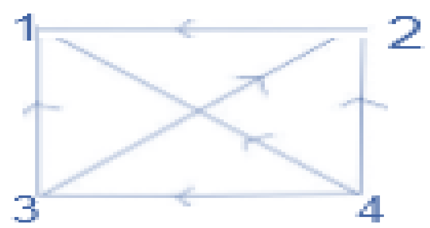
Example:

Let X = {1, 2, 3, 4} and R={(x, y) | x > y}. Draw the graph of R and also give its matrix.

Solution:

R = {(4, 1), (4, 3), (4, 2), (3, 1), (3, 2), (2, 1)}.

The graph of R and the matrix of R are



Graph of R

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

compatibility relation

Definition:

- A relation R in S is said to be a compatibility relation if it is reflexive and symmetric.
- Clearly, all equivalence relations are compatibility relations.
- A compatibility relation is sometimes denoted by ≈.

Example: Let X = {ball, bed, dog, let, egg}, and let the relation R be given by R = {(x, y) | x, y ∈ X ∧ xRy if x and y contain some common letter}.

- Then R is a compatibility relation, and x, y are called compatible if xRy.
- Note: ball≈bed, bed≈egg. But ball≉egg. Thus ≈ is not transitive.

Composition of binary relations

- Let R be a relation from X to Y and S be a relation from Y to Z. Then the relation R o S is given by
- R o S = {(x, z) / x∈X ∧ z ∈ Z ∧ y ∈ Y such that (x, y) ∈ R ∧ (y, z) ∈S)} is called the composite relation of R and S.
- The operation of obtaining R o S is called the composition of relations.
- **Example1:** Let R = {(1, 2), (3, 4), (2, 2)} and

  S = {(4, 2), (2, 5), (3, 1),(1,3)} Then

  R o S = {(1, 5), (3, 2), (2, 5)} and

  S o R = {(4, 2), (3, 2), (1, 4)}

  It is to be noted that R o S ≠ S o R.

  Also Ro(S o T) = (R o S) o T = R o S o T

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Example2: Let R = {(1, 2), (3, 4), (2, 2)} and S = {(4, 2), (2, 5), (3, 1), (1, 3)}. Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$, and $(R \circ R) \circ R$.

Solution:

Given R = {(1, 2), (3, 4), (2, 2)} and S = {(4, 2), (2, 5), (3, 1), (1, 3)}.

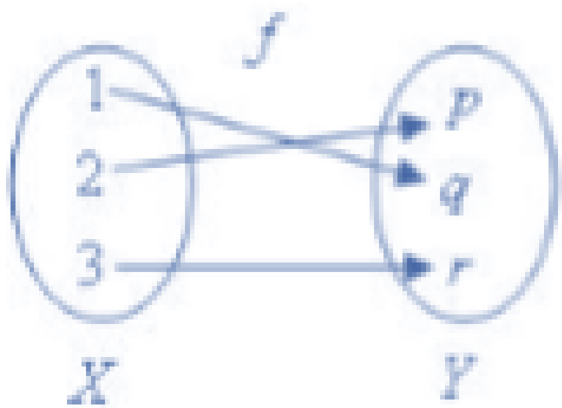
- $R \circ S = \{(1, 5), (3, 2), (2, 5)\}$
- $S \circ R = \{(4, 2), (3, 2), (1, 4)\} \neq R \circ S$
- $(R \circ S) \circ R = \{(3, 2)\}$
- $R \circ (S \circ R) = \{(3, 2)\} = (R \circ S) \circ R$
- $R \circ R = \{(1, 2), (2, 2)\}$
- $R \circ R \circ S = \{(4, 5), (3, 3), (1, 1)\}$

Functions

- A function is a special case of relation.

Definition: Let X and Y be any two sets. A relation f from X to Y is called a function if for every $x \in X$, there is a unique element $y \in Y$ such that $(x, y) \in f$.

Example: Let X = {1, 2, 3}, Y = {p, q, r} and f = {(1, p), (2, q), (3, r)} then f(1) = p, f(2) = q, f(3) = r. Clearly f is a function from X to Y .



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- **Domain and Range of a Function:** If $f : X \rightarrow Y$ is a function, then X is called the **Domain** of f and the set Y is called the **codomain** of f.
- The range of f is defined as the set of all images under f. It is denoted by $f(X) = \{y \mid \text{ for some } x \text{ in } X, f(x) = y\}$ and is called the image of X in Y . The Range f is also denoted by R_f .
- **Example:** If the function f is defined by $f(x)=x^2 + 1$ on the set $\{-2, -1, 0, 1, 2\}$, find the range of f.

Solution: $f(-2) = (-2)^2 + 1 = 5$ $f(-1) = (-1)^2 + 1 = 2$

$f(0) = 0 + 1 = 1$

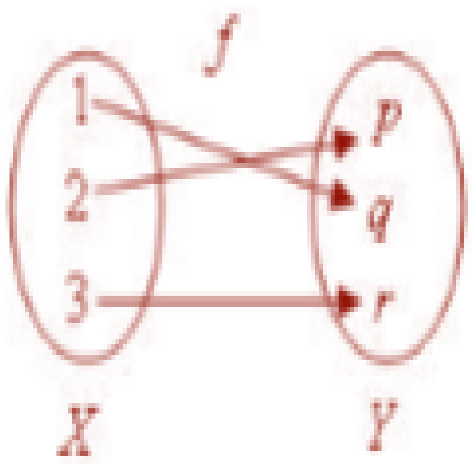
$f(1) = 1 + 1 = 2$

$f(2) = 4 + 1 = 5$

Therefore, the range of f = {1, 2, 5}.

Types of Functions

- **<u>One-to-one(Injective):</u>** A mapping $f : X \rightarrow Y$ is called one-to-one if distinct elements of X are mapped into distinct elements of Y ,
    - i.e., f is one-to-one if $x1 \neq x2 \Rightarrow f(x1) \neq f(x2)$ or equivalently $f(x1) = f(x2) \Rightarrow x1 = x2$ for $x1, x2 \in X$.



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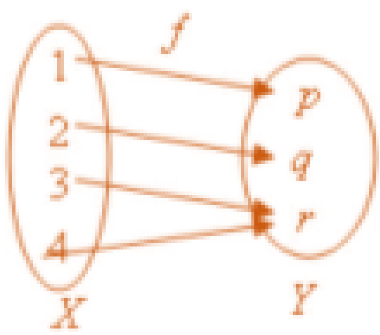
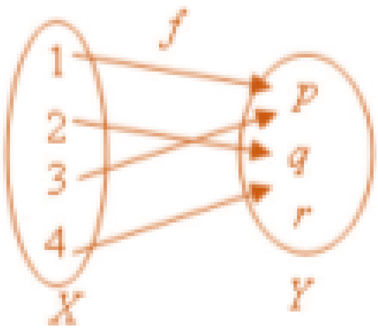
- **Onto(Surjective):** A mapping $f : X \rightarrow Y$ is called onto if the range set $R_f = Y$.
- If $f : X \rightarrow Y$ is onto, then each element of Y is f-image of atleast one element of X.
- i.e., $\{f(x) : x \in X\} = Y$.
- If f is not onto, then it is said to be into.



Surjective



Not Surjective

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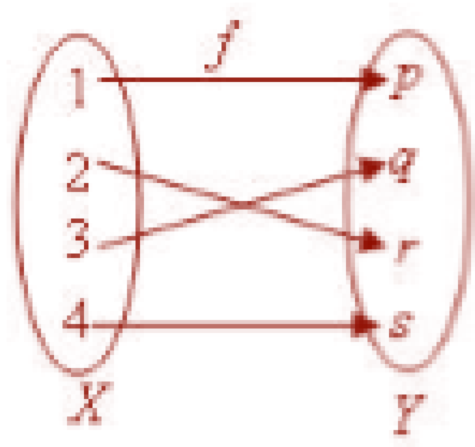
- **Bijection or One-to-One, Onto:**

  A mapping f : X → Y is called one-to-one, onto or bijective if it is both one-to-one and onto. Such a mapping is also called a one-to-one correspondence between X and Y .



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- **Identity function:**

Let X be any set and f be a function such that $f : X \to X$ is defined by $f(x) = x$ for all $x \in X$. Then, f is called the identity function or identity transformation on X. It can be denoted by I or Ix.

- **Inverse Functions:**

A function $f : X \to Y$ is aid to be invertible of its inverse function f^{-1} is also function from the range of f into X.

Note: A function $f : X \to Y$ is invertible $\Leftrightarrow$ f is one-to-one and onto.

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- **Constant Functions:**

  A mapping f: R→ b is called a constant mapping if, for all a∈A, f (a) = b, a fixed element.

  For example f: Z→Z given by f(x) = 0, for all x ∈Z is a constant mapping.

Composition of Functions:

- Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the composition of f and g denoted by $g \circ f$, is the function from X to Z defined as $(g \circ f)(x) = g(f(x))$, for all $x \in X$.

Example 1: Let X = {1, 2, 3}, Y = {p, q} and Z = {a, b}. Also let $f : X \to Y$ be f = {(1, p), (2, q), (3, q)} and $g : Y \to Z$ be given by g = {(p, b), (q, b)}. Find $g \circ f$.

Solution:

$g \circ f$ = {(1, b), (2, b), (3, b)}.

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- **Example2:** Let X = {1, 2, 3} and f, g, h and s be the functions from X to X given by f = {(1, 2), (2, 3), (3, 1)} g = {(1, 2), (2, 1), (3, 3)} h = {(1, 1), (2, 2), (3, 1)}

  s = {(1, 1), (2, 2), (3, 3)} Find $f \circ f$; $g \circ f$; $f \circ h \circ g$; $s \circ g$; $g \circ s$; $s \circ s$; and $f \circ s$.

Solution:

$f \circ g = \{(1, 3), (2, 2), (3, 1)\}$

$g \circ f = \{(1, 1), (2, 3), (3, 2)\} \neq f \circ g$

$f \circ h \circ g = f \circ (h \circ g) = f \circ \{(1, 2), (2, 1), (3, 1)\} = \{(1, 3), (2, 2), (3, 2)\}$

$s \circ g = \{(1, 2), (2, 1), (3, 3)\} = g$

$g \circ s = \{(1, 2), (2, 1), (3, 3)\}$ $\quad \therefore s \circ g = g \circ s = g$

$s \circ s = \{(1, 1), (2, 2), (3, 3)\} = s$

$f \circ s = \{(1, 2), (2, 3), (3, 1)\}$ Thus, $s \circ s = s$, $f \circ g \neq g \circ f$,

$s \circ g = g \circ s = g$ and $h \circ s = s \circ h = h$.

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- **Example3:** Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in R$, where R is the set of real numbers. Find $g \circ f$; $f \circ g$; $f \circ f$; $g \circ g$; $f \circ h$; $h \circ g$; $h \circ f$; and $f \circ h \circ g$.

Solution: $f : R \to R$ is defined by $f(x) = x + 2$ $f: R \to R$ is defined by $g(x) = x - 2$ $h : R \to R$ is defined by $h(x) = 3x$

$g \circ f : R \to R$ Let $x \in R$. Thus, we can write $(g \circ f)(x) = g(f(x)) = g(x + 2) = x + 2 - 2 = x$

$\therefore (g \circ f)(x) = \{(x, x) \mid x \in R\}$

$(f \circ g)(x) = f(g(x)) = f(x - 2) = (x - 2) + 2 = x$

$\therefore f \circ g = \{(x, x) \mid x \in R\}$

$(f \circ f)(x) = f(f(x)) = f(x + 2) = x + 2 + 2 = x + 4$

$\therefore f \circ f = \{(x, x + 4) \mid x \in R\}$

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$(g \circ g)(x) = g(g(x)) = g(x - 2) = x - 2 - 2 = x - 4$

$\Rightarrow g \circ g = \{(x, x - 4) \mid x \in R\}$

$(f \circ h)(x) = f(h(x)) = f(3x) = 3x + 2$

$\therefore f \circ h = \{(x, 3x + 2) \mid x \in R\}$

$(h \circ g)(x) = h(g(x)) = h(x - 2) = 3(x - 2) = 3x - 6$

$\therefore h \circ g = \{(x, 3x - 6) \mid x \in R\}$

$(h \circ f)(x) = h(f(x)) = h(x + 2) = 3(x + 2) = 3x + 6$

$\therefore h \circ f = \{(x, 3x + 6) \mid x \in R\}$

$(f \circ h \circ g)(x) = [f \circ (h \circ g)](x)\ f(h \circ g(x)) = f(3x - 6) = 3x - 6 + 2 = 3x - 4$

$\therefore f \circ h \circ g = \{(x, 3x - 4) \mid x \in R\}$.

Thank You