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Vertex arboricity of integer distance graph $G(D_{m,k})$ [☆]

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ABSTRACT

Let D be a subset of the positive integers. The *distance graph* $G(\mathbb{Z}, D)$ has all integers as its vertices and two vertices x and y are adjacent if and only if $|x - y| \in D$, where the set D is called *distance set*. The vertex arboricity $va(G)$ of a graph G is the minimum number of subsets into which vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. In this paper, the vertex arboricity of graphs $G(\mathbb{Z}, D_{m,k})$ are studied, where $D_{m,k} = \{1, 2, \ldots, m\} \setminus \{k\}$. In particular, $va(G(D_{m,1})) = \lceil \frac{m+3}{4} \rceil$ for any integer $m \geq 5$; $va(G(D_{m,2})) = \lceil \frac{m+1}{4} \rceil + 1$ for $m = 8l + j \geq 6$ and $j \neq 7$, and $\lceil \frac{m}{4} \rceil + 1 \leq va(G(D_{m,2})) \leq \lceil \frac{m}{4} \rceil + 2$ for $m = 8l + 7$.

1. Introduction

In this paper, $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of all real numbers and all integers, respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x; $\lceil x \rceil$ denotes the least integer not less than x; we use $[m, n]$ for the set of the integers from m to n ($m \leq n$) and $[m, n] = \emptyset$ if $m > n$. $|S|$ denotes the cardinality of a set S ($|S| = +\infty$ means that S is an infinite set).

Coloring in graphs has been one of the most fascinating and well-studied topics in graph theory. Its root goes back to the Four Color Conjecture and more recently, it was motivated by such application problems as the frequency assignment problem (i.e., $L(2, 1)$-labeling), the control of traffic signals (i.e., circular coloring) and other problems from wide range of industrial areas. A vertex-coloring (or edge-coloring) can be viewed as a function from V (or E) to $\mathbb{Z}$. More precisely, a *k-coloring* of a graph G is a mapping f from $V(G)$ to $[1, k]$. Given a k-coloring, let V_i denote the set of all vertices of G colored with i, and $\langle V_i \rangle$ denote the subgraph induced by V_i in G. If V_i is an independent set for every $1 \leq i \leq k$, then f is called a *proper k-coloring*. The *chromatic number* $\chi(G)$ of a graph G is the minimum integer k for which G has a proper k-coloring. If V_i induces a subgraph whose connected components are trees, then f is called a *tree k-coloring*. The *vertex arboricity* of a graph G, denoted by $va(G)$, is the minimum integer k for which G has a tree k-coloring. In other words, the vertex arboricity $va(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. If V_i induces a subgraph whose connected components are paths, then f is called a *path k-coloring*. The *vertex linear arboricity* of a graph G, denoted by $vla(G)$, is the minimum number k for which G has a path k-coloring. Clearly, $\chi(G) \geq vla(G) \geq va(G)$ for any graph G.

Since the introduction of vertex arboricity, it has been investigated widely by many researchers for various properties and its links to other graphic parameters. For instance, Kronk et al. [7] proved that $va(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G. Catlin and Lai [2] showed that when G is a graph that is neither a cycle nor a clique, $va(G) \leq \lceil \frac{\Delta(G)}{2} \rceil$. Škrekovski [9] proved that locally

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planar graphs have the vertex arboricity at most 3 and that triangle-free locally planar graphs have the vertex arboricity at most 2. Jørgensen [4] studied $K_{4,4}$-minor free graphs and showed that the vertex arboricity is at most 4. In this paper, we study the vertex arboricity of a family of infinite graphs, integer distance graphs, and determine the exact value $va(G)$ of such graphs.

Let S be a subset of all real numbers and D a set of positive real numbers. Then *distance graph* $G(S, D)$ has the vertex set S and two real numbers x and y are adjacent if and only if $|x-y| \in D$, where the set D is called *distance set*. In particular, if all elements of D are positive integers and $S = \mathbb{Z}$, the graph $G(\mathbb{Z}, D)$, or $G(D)$ in short, is called *integer distance graph*. The distance graphs were introduced by Eggleton et al. [3] in 1985 to study the chromatic number. They proved that $\chi(G(\mathbb{R}, D)) = n+2$, where D is an interval between 1 and δ, and n satisfies $1 \le n < \delta \le n+1$. They also partially determined the values of $\chi(G(D_{m,k}))$, where $D_{m,k} = [1, m] \setminus \{k\}$. The complete solution to $\chi(G(D_{m,k}))$ is provided by Chang, Liu and Zhu in [1]. In [11, 12], Zuo et al. examined the vertex linear arboricity of the distance graph $G(\mathbb{R}, D)$ with an interval D and the integer distance graph $G(D_{m,k})$, respectively. In [13], Zuo, Yu and Wu studied that the vertex arboricity of the distance graph $G(\mathbb{R}, D)$ with an interval D. The interested reader is referred to [3,5,6,8,10–13] for more details. More recently, integer distance graphs have found applications in gene sequencing, sequential series, on-line computing, etc. and gained more attention for its properties.

In this paper, we study the vertex arboricity of $G(D_{m,k})$ for $D_{m,k} = [1, m] \setminus \{k\}$ and determine the exact values for $k = 1, 2$, and also provide upper and lower bounds for general k.

2. Vertex arboricity of $G(D_{m,1})$

Clearly, $va(G(D)) = 1$ if $|D| = 1$. If $|D| \ge 2$, then $va(G(D)) \ge 2$ since $G(D)$ contains a cycle with vertices $a, 2a, \ldots, ba, b(a-1), \ldots, b, 0$ for $a, b \in D$ and $a \ne b$. It is obvious that $va(G(D_2)) \le va(G(D_1))$ if $D_2 \subseteq D_1$.

Lemma 2.1. (1) *For any finite distance set D, $va(G(D)) \le \lceil \frac{|D|+1}{2} \rceil$ and the bound is sharp;*
(2) *For any positive integer k, $va(G(D)) \le k$ if there is at most one multiple of k in D.*

Proof. (1) Let $k = \lceil \frac{|D|+1}{2} \rceil$. We color the vertices of $G(D)$ recursively with colors $[1, k]$ as follows. First, let $f(0) = 1$. Assume that all $f(j)$ are colored for some i and $-i \le j \le i$. Let A be the set of colors appearing twice in vertices of $\{j \mid -i \le j \le i \text{ and } i+1-j \in D\}$. Then $|A| \le \left\lfloor \frac{|D|}{2} \right\rfloor$ and we assign $f(i+1)$ to any value of $[1, k] \setminus A$ (in fact, we may choose $f(i+1) = \min\{t \mid t \in [1, k] \setminus A\}$). Similarly, let B be the set of colors appearing twice in vertices of $\{j \mid -i \le j \le i+1 \text{ and } j+i+1 \in D\}$. Then $|B| \le \left\lfloor \frac{|D|}{2} \right\rfloor$. So we assign $f(-i-1)$ to any value of $[1, k] \setminus B$ (we may choose $f(-i-1) = \min\{t \mid t \in [1, k] \setminus B\}$).

Now we see f is a tree $\left\lceil \frac{|D|+1}{2} \right\rceil$-coloring. Otherwise, if there is a cycle induced by the vertices receiving the same color α, then there exists an integer i such that $f(i+1) \in A$ or $f(-i-1) \in B$, a contradiction. Hence, $va(G(D)) \le \left\lceil \frac{|D|+1}{2} \right\rceil$.

This bound is sharp. For example, for any positive integer m, let $D = [1, m]$, then $va(G(D)) \le \lceil \frac{m+1}{2} \rceil = \lceil \frac{|D|+1}{2} \rceil$ and thus $va(G(D)) = \lceil \frac{|D|+1}{2} \rceil$ since vertices $0, 1, 2, \ldots, m$ induce a complete graph K_{m+1}.

(2) Let $f(n) \equiv n \pmod k$. Then the subgraph induced by vertices in $\{v \mid f(v) = i\}$ is a forest for each $i \in [0, k-1]$, that is, f is a tree coloring. Thus $va(G(D)) \le k$. □

Let $D_{m,k} = [1, m] \setminus \{k\}$ for any positive integers m, k with $m > k$. Before proceeding to the main results, we present a lemma which is handy in the proofs of later theorems.

Lemma 2.2. *For an integer distance graph $G(D_{m,k})$ and a fixed integer i, if $n_0 \ge m + 2k + 1$, then each of the following vertex subsets*

$$V_i = \{i + sn_0, i + sn_0 + k, i + sn_0 + 2k, i + sn_0 + 3k \mid s \in \mathbb{Z}\},$$
$$V_i' = \{i + sn_0, i + sn_0 + 1 \mid s \in \mathbb{Z}\},$$
$$V_i'' = \left\{i + sn_0, i + sn_0 + \left\lceil \frac{k}{2} \right\rceil, i + sn_0 + k \mid s \in \mathbb{Z}\right\}$$

induces a forest.

Proof. We only deal with the first set and other cases can be proved similarly.

Clearly, the vertices $i+sn_0, i+sn_0+k, i+sn_0+2k, i+sn_0+3k$ induce a path for any integer s. Since $n_0 \ge m+2k+1$, the vertices $i+sn_0$, $i+sn_0+k$ and $i+sn_0+2k$ are not adjacent to each of the vertices $i+(s+1)n_0$, $i+(s+1)n_0+k$, $i+(s+1)n_0+2k$ and $i+(s+1)n_0+3k$, and the vertex $i+sn_0+3k$ is not adjacent to each of the vertices $i+(s+1)n_0+k$, $i+(s+1)n_0+2k$ and $i+(s+1)n_0+3k$. Hence the lemma holds. □

Next, we study vertex arboricity of $G(D_{m,k})$ for case $k = 1$.

Theorem 2.1. *For any integer $m \ge 3$, $va(G(D_{m,1})) = \lceil \frac{m+3}{4} \rceil$.*

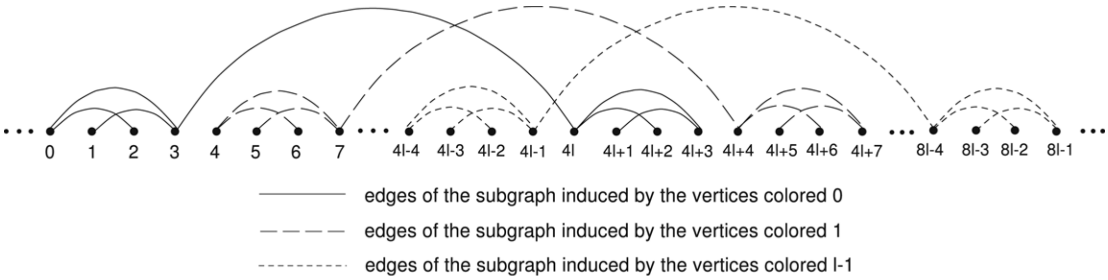

Fig. 1. Tree $\lceil \frac{m+3}{4} \rceil$-coloring for $m = 4q + 1 \geq 5$.

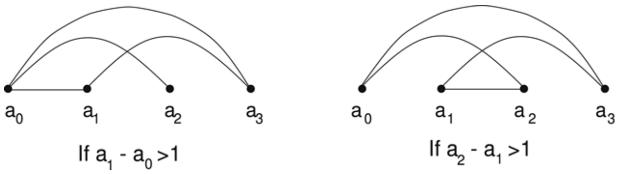

Fig. 2. $a_3 - a_0 \leq m$.

Proof. For $3 \leq m \leq 4$, by Lemma 2.1, $va(G(D_{m,1})) = 2$. So we assume $m \geq 5$.

Firstly, we construct a tree coloring f in $G(D_{m,1})$ as follows. Let $l = \lceil \frac{m+3}{4} \rceil$. Define $f(4t+i) = t$, for $0 \leq t < l$ and $0 \leq i \leq 3$; and other vertices are colored periodically, that is, $f(n + 4ls) = f(n)$ for all $n, s \in \mathbb{Z}$. By Lemma 2.2,

$$V_t = \cup_{k\in\mathbb{Z}}[4kl + 4t, 4kl + 4t + 3]$$

induces an acyclic subgraph for each $0 \leq t < l$. Thus f is a tree coloring of $G(D_{m,1})$ and $va(G(D_{m,1})) \leq \lceil \frac{m+3}{4} \rceil$ (see Fig. 1).

Secondly, we show that $va(G(D_{m,1})) \geq \lceil \frac{m+3}{4} \rceil$. Assume, to the contrary, $G(D_{m,1})$ has a tree $\lceil \frac{m-1}{4} \rceil$-coloring f. Let H be a subgraph of $G(D_{m,1})$ induced by vertices $[0, m+2]$. Then f is also a tree $\lceil \frac{m-1}{4} \rceil$-coloring of H. Note that $|V(H)| = m+3$. There are at least five vertices in H, say $0 \leq a_0 < a_1 < \cdots < a_4 \leq m+2$, receiving the same color α.

Claim 1. If $a_3 - a_0 \leq m$, then $a_3 = a_2 + 1 = a_1 + 2 = a_0 + 3$.

Clearly, $a_0a_2, a_0a_3, a_1a_3 \in E(H)$ in this case. If $a_1 - a_0 > 1$, then $a_0a_1 \in E(H)$ and a_0, a_1, a_3 induce a triangle (see Fig. 2), a contradiction. So $a_1 - a_0 = 1$. If $a_2 - a_1 > 1$, then $a_1a_2 \in E(H)$, so a_0, a_2, a_1, a_3 induce a cycle of length 4, a contradiction. Hence $a_2 - a_1 = 1$. It is similar to see that $a_3 - a_2 = 1$.

Claim 2. $\min\{a_3 - a_0, a_4 - a_1\} > m$.

If $a_3 - a_0 \leq m$, by Claim 1, then $a_3 = a_2 + 1 = a_1 + 2 = a_0 + 3$, and $a_0a_2, a_0a_3, a_1a_3 \in E(H)$. Since $a_4 \leq m+2$ and $a_2 \geq 2$, we have $a_2a_4 \in E(H)$. So $a_1a_4 \notin E(H)$ (otherwise, a_0, a_3, a_1, a_4, a_2 form a cycle of length 5, a contradiction), that is, $a_4 - a_1 = m+1$, $a_4 = m+2$, $a_1 = 1$, $a_3 = 3$. Thus, $a_3a_4 \in E(H)$ and then a_0, a_2, a_3, a_4 induce a cycle of length 4, a contradiction. Therefore $a_3 - a_0 > m$. Similarly, $a_4 - a_1 > m$.

Claim 3. $a_0 = 0$, $a_1 = 1$, $a_3 = m+1$, $a_4 = m+2$ and $a_2 \in \{2, m\}$.

It is clear that $a_0 = 0$, $a_1 = 1$, $a_3 = m+1$, $a_4 = m+2$ and $a_1a_3 \in E(H)$ by Claim 2. Next, we see that $a_2 \in \{2, m\}$. Otherwise, if $2 < a_2 < m$, then $a_1a_2, a_2a_3 \in E(H)$ and thus a_1, a_2, a_3 induce a triangle, a contradiction.

Without loss of generality, assume that $a_2 = 2$.

Claim 4. $m \equiv 2 \pmod 4$.

Otherwise, we have $m + 3 \not\equiv 1 \pmod 4$ and then there exists another color β used on five vertices $3 \leq b_0 < b_1 < \cdots < b_4 \leq m$. Thus $b_0b_2, b_2b_4, b_0b_4 \in E(H)$, i.e., b_0, b_2, b_4 induce a triangle, a contradiction.

The last claim implies that except α, any other color is used on only four vertices in H, and these four vertices must be consecutive. That is, vertices 3, 4, 5 and 6 receive one color, vertices 7, 8, 9 and 10 receive another color and so on.

Now we analyze the coloring of vertex $m+4$ of $G(D_{m,1})$. Suppose $f(m+4) = \beta \neq \alpha$, then there exists l, where $3 \leq l \leq m-3$, such that $f(l) = f(l+1) = f(l+2) = f(l+3) = \beta$. Since $m+4$ and l are both adjacent to $l+2, l+3$, we see that $l, l+2, l+3$ and $m+4$ induce a 4-cycle, a contradiction. So $f(m+4) = \alpha$. But, then vertices $2, m+1, m+4$ and $m+2$ induce a cycle of length 4, a contradiction again.

Therefore $va(G(D_{m,1})) \geq \lceil \frac{m+3}{4} \rceil$. □

Next, we present an algorithm for finding a tree coloring of $G(D_{m,1})$.

If $m = 2$, assign 0 to all vertices; if $3 \leq m \leq 4$, assign 0 to vertices x, where $x \pmod 8 \in [0, 3]$ and assign 1 to vertices y, where $y \pmod 8 \in [4, 7]$. For $m \geq 5$ and $l = \lceil \frac{m+3}{4} \rceil$, we have the following algorithm.

Algorithm. $A(m, 1)$. For a vertex x, if $x = 4t + r$ for $0 \leq t < l$ and $0 \leq r < 4$, then x is colored with t (i.e., $f(x) = t$); otherwise, $x = 4ls + x'$ for some $0 \leq x' < 4l$ and $s \in \mathbb{Z}$, then x is colored with $f(x')$. Continue this process until every vertex receives a color.

3. Vertex arboricity of $G(D_{m,2})$

In this section, we study $va(G(D_{m,k}))$ for the case $k = 2$. From Lemma 2.1, we have $va(G(D_{3,2})) = va(G(D_{4,2})) = va(G(D_{5,2})) = 2$. So we assume $m \geq 6$.

We summarize the basic tactics used in the proof of the main result as three lemmas.

Lemma 3.1. *Suppose there are three vertices $b_1 < b_2 < b_3$ ($b_i \in \mathbb{Z}$, $i = 1, 2, 3$) receiving the same color in $G(D_{m,2})$.*

(1) *if there is a (b_1, b_2)-path in $G(D_{m,2})$, then $b_3 \in \{b_1 + 2, b_2 + 2\}$ or $b_3 \geq b_1 + (m + 1)$;*
(2) *if there is a (b_1, b_3)-path in $G(D_{m,2})$ and $b_3 - b_1 \leq m$, then $b_2 \in \{b_1 + 2, b_3 - 2\}$;*
(3) *if there is a (b_2, b_3)-path in $G(D_{m,2})$, then $b_1 \in \{b_2 - 2, b_3 - 2\}$ or $b_1 \leq b_3 - (m + 1)$.*

Proof. (1) Otherwise, if $b_3 \notin \{b_1 + 2, b_2 + 2\}$ and $b_3 - b_1 \leq m$, then $b_1b_3, b_2b_3 \in E(H)$ and thus (b_1, b_2)-path and two edges b_1b_3, b_2b_3 form a cycle, a contradiction.

(2) and (3) can be proved similarly. □

Lemma 3.2. *Let H_1 and H_2 be subgraphs of $G(D)$ induced by vertices $[c, l]$ ($c < l$, $c, l \in \mathbb{Z}$) and vertices $[c+s, l+s]$ (for any $s \in \mathbb{Z}$), respectively. Then H_1 has a tree n-coloring if and only if H_2 has a tree n-coloring.*

Proof. Since $ij \in E(H_1)$ ($i, j \in [c, l]$) if and only if $(s + i)(s + j) \in E(H_2)$, H_1 and H_2 are isomorphic and the conclusion follows. □

For the convenience of arguments, we introduce a new term. If four vertices $v, v + 2, v + 4, v + 6$ receive a color β, then such a set $\{v, v + 2, v + 4, v + 6\}$ is called an *F-type set associated with β and v* and denoted by V_{β_v}. If there is no confusion arising, we often call it F-type set, in short.

Lemma 3.3. *If V_{β_v} is an F-type set associated with β and v, where $j_0 \leq v \leq m - 2$ for a fixed positive integer j_0, then $m + i \notin V_{\beta_v}$ for any i with $5 \leq i \leq j_0 + 4$.*

Proof. Assume, to the contrary, that $m+i \in V_{\beta_v}$ for some i with $5 \leq i \leq j_0+4$. Since v is adjacent to $v+4$ and $v+6$, by taking $b_1 = v+4$, $b_2 = v+6$ and $b_3 = m+i$ in Lemma 3.1 (1), we have $m+i = (v+6)+2$ or $m+i \geq v+4+(m+1) \geq m+j_0+5$. However, $m + i \leq m + j_0 + 4$ by hypothesis, thus we have $m + i = (v + 6) + 2$, i.e., $m + i - (v + 4) = 4$. So $v(m + i), (v + 4)(m + i) \in E(H)$ and then vertices $v, v + 4$ and $m + i$ induce a triangle, a contradiction. □

Theorem 3.1. *Let $m = 8l + j \geq 6$, where $0 < j \leq 8$. Then*

$$va(G(D_{m,2})) = \left\lceil \frac{m+1}{4} \right\rceil + 1 \quad \text{for } j \neq 7$$

and

$$\left\lceil \frac{m}{4} \right\rceil + 1 \leq va(G(D_{m,2})) \leq \left\lceil \frac{m}{4} \right\rceil + 2 \quad \text{for } j = 7.$$

Proof. Firstly, we show the upper bound

$$va(G(D_{m,2})) \leq \begin{cases} \left\lceil \dfrac{m+1}{4} \right\rceil + 1 & \text{for } j \neq 7, \\ \left\lceil \dfrac{m}{4} \right\rceil + 2 & \text{for } j = 7. \end{cases}$$

We define a tree coloring of $G(D_{m,2})$ periodically.

For $1 \leq j \leq 3$, let $f_1(8t + i) = f_1(8t + i + 2) = f_1(8t + i + 4) = f_1(8t + i + 6) = 2t + i$ for $0 \leq t \leq l$ and $i = 0, 1$, and $f_1(n + 8(l + 1)s) = f_1(n)$ for all $n, s \in \mathbb{Z}$. Since each $V^{(1)}_{t,i} = \{8(l + 1)s + 8t + i + 2r \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ induces a forest by Lemma 2.2, f_1 is a tree coloring (see Fig. 3) and thus $va(G(D_{m,2})) \leq 2\lceil \frac{m}{8} \rceil = \lceil \frac{m+1}{4} \rceil + 1$.

For $4 \leq j \leq 6$, let $f_2(8t + i) = f_2(8t + i + 2) = f_2(8t + i + 4) = f_2(8t + i + 6) = 2t + i$ for $0 \leq t \leq l$ and $0 \leq i \leq 1$, $f_2(8(l + 1)) = f_2(8(l + 1) + 1) = f_2(8(l + 1) + 2) = 2(l + 1)$ and $f_2(n + 8(l + 1) + 3) = f_2(n)$ for all $n \in \mathbb{Z}$. Since each of $V^{(2)}_{t,i} = \{(8(l + 1) + 3)s + 8t + i + 2r \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ and $V^{(2)}_{l+1} = \{(8(l + 1) + 3)s + 8(l + 1) + r \mid s \in \mathbb{Z}, r \in [0, 2]\}$ induces a forest by Lemma 2.2, f_2 is a tree coloring and thus $va(G(D_{m,2})) \leq 2\lceil \frac{m}{8} \rceil + 1$, or $va(G(D_{m,2})) \leq \lceil \frac{m+1}{4} \rceil + 1$ for $m = 8l + j$ with $4 \leq j \leq 6$.

For $7 \leq j \leq 8$, let $f_3(8t + i) = f_3(8t + i + 2) = f_3(8t + i + 4) = f_3(8t + i + 6) = 2t + i$ for $0 \leq t \leq l + 1$ and $0 \leq i \leq 1$, and $f_3(8(l+2)s+n) = f_3(n)$ for all $n, s \in \mathbb{Z}$. Since each $V^{(3)}_{t,i} = \{8(l+2)s+8t+i+2r \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ induces a forest by Lemma 2.2, f_3 is a tree coloring and thus $va(G(D_{m,2})) \leq 2(\lceil \frac{m}{8} \rceil + 1) = \lceil \frac{m}{4} \rceil + 2$ for $j = 7$ and $va(G(D_{m,2})) \leq 2(\lceil \frac{m}{8} \rceil + 1) = \lceil \frac{m+1}{4} \rceil + 1$ for $j = 8$.

Hence, the upper bound is confirmed.

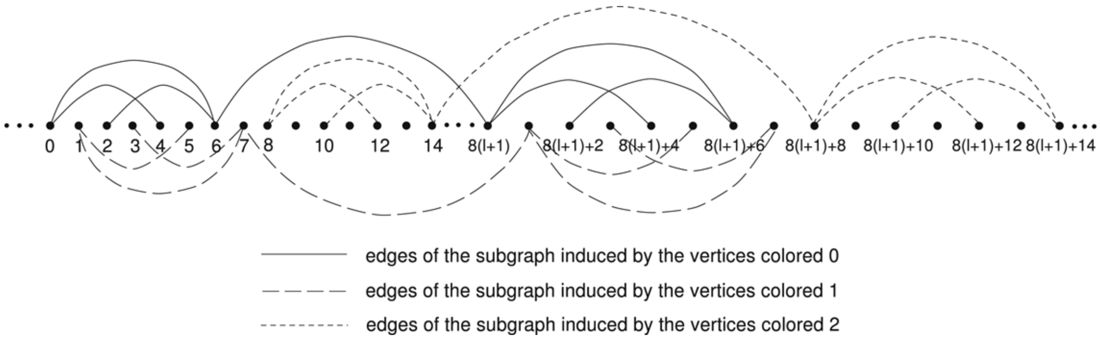

Fig. 3. Tree ($\lceil \frac{m+1}{4} \rceil + 1$)-coloring for $m = 8l + j (1 \le j \le 3)$.

Next, we show the lower bound

$$va(G(D_{m,2})) \ge \left\lceil \frac{m+1}{4} \right\rceil + 1 \quad \text{for } m = 4q + j \ge 6.$$

First, we claim $va(G(D_{m,2})) \ge \lceil \frac{m+1}{4} \rceil + 1$ for $m = 4q \ge 8$.

Assume, to the contrary, that $va(G(D_{m,2})) \le \lceil \frac{m+1}{4} \rceil = \lceil \frac{m}{4} \rceil + 1 = q + 1$, then $G(D_{m,2})$ has a tree $(q+1)-$coloring f. Let H be a subgraph induced by vertex subset $[0, m+4]$. Then f is also a tree coloring of H. Note that $|V(H)| = m + 5$. There exist at least five vertices in H, say $0 \le a_0 < a_1 < \cdots < a_4 \le m + 4$, receiving the same color α.

Claim 1. (1) If $a_0 + 2 \le a_1 < a_2 \le a_3 - 2$ and $a_3 - a_0 \le m + 3$, then $a_1 = a_0 + 2$ or $a_2 = a_3 - 2$; (2) if $a_3 - a_0 \le m + 1$, then at least two equalities in $\{a_{i+1} - a_i = 2 \mid i \in [0, 2]\}$ hold; moreover, if $a_3 - a_0 = m + 1$, then exactly two equalities in $\{a_{i+1} - a_i = 2 \mid i \in [0, 2]\}$ hold; (3) if $a_3 - a_0 \le m$, then $a_{i+1} - a_i = 2$ for all $i \in [0, 2]$.

(1) Otherwise, if $a_3 - a_0 \le m+3$ but $a_0 + 3 \le a_1 < a_2 \le a_3 - 3$, then $3 \le a_3 - a_1 \le a_3 - (a_0 + 3) \le m$ and thus $a_1a_3 \in E(H)$. Similarly, $a_0a_1, a_0a_2, a_2a_3 \in E(H)$ and thus a_0, a_1, a_2, a_3 induce a 4-cycle, a contradiction.

(2) If $a_{i+1} - a_i \ne 2$ for each $i \in [0, 2]$, then $a_0a_1, a_1a_2, a_2a_3 \in E(H)$. Thus $a_0a_2, a_1a_3 \notin E(H)$, i.e., $a_2 - a_0 = a_3 - a_1 = 2$, and it implies that $a_3 - a_0 = 3$ and $a_0a_3 \in E(H)$. Hence a_0, a_1, a_2, a_3 induce a 4-cycle, a contradiction.

Suppose that only one equality in $\{a_{i+1} - a_i = 2 \mid i \in [0, 2]\}$ holds. If $a_1 - a_0 = 2$, then $a_2 - a_1 \ne 2$, $a_3 - a_2 \ne 2$ and $a_1a_2, a_2a_3 \in E(H)$. Moreover, $a_3 - a_1 = (a_3 - a_0) - (a_1 - a_0) \le m - 1$ and then $a_1a_3 \in E(H)$, thus a_1, a_2, a_3 induce a triangle; similarly, if $a_3 - a_2 = 2$, then a_0, a_1, a_2 induce a triangle; if $a_2 - a_1 = 2$, then a_0, a_1, a_3, a_2 induce a 4-cycle. Hence at least two equalities hold.

Moreover, suppose $a_3 - a_0 = m + 1$. If all three equalities hold, then $a_3 - a_0 = 6 = m + 1$ which contradicts $m \ge 8$. Hence exactly two equalities in $\{a_{i+1} - a_i = 2 \mid i \in [0, 2]\}$ hold.

(3) From (2), at least two equalities in $\{a_{i+1} - a_i = 2 \mid i \in [0, 2]\}$ hold. Without loss of generality, say $a_3 - a_2 = a_1 - a_0 = 2$, then $a_0a_3, a_0a_2, a_1a_3 \in E(H)$, so $a_1a_2 \notin E(H)$, that is, $a_2 - a_1 = 2$.

Claim 2. $\min\{a_3 - a_0, a_4 - a_1\} > m$.

We need only to show that $a_3 - a_0 > m$ and $a_4 - a_1 > m$. Assume, to the contrary, that $a_3 - a_0 \le m$, then $a_3 = a_2 + 2 = a_1 + 4 = a_0 + 6$ by Claim 1(3), and thus there is a (a_2, a_3)-path in H. By taking $b_i = a_{i+1}$ $(i = 1, 2, 3)$ in Lemma 3.1(1), we have $a_4 \ge a_2 + (m+1) = a_0 + (m+5) \ge m + 5$, or $a_4 = a_3 + 2$ and thus a_0, a_2, a_4, a_1, a_3 induce a 5-cycle, a contradiction. Similarly, we can show that $a_4 - a_1 > m$.

As a consequence of Claim 2, the range of some a_i's location on the integer axis can be determined, e.g., $0 \le a_0 \le a_3 - (m+1) \le 2$ or $a_0 \in [0, 2]$, $m + 1 \le a_0 + (m+1) \le a_3 \le m + 3$ or $a_3 \in [m+1, m+3]$ and similarly $a_1 \in [1, 3]$, $a_4 \in [m+2, m+4]$. The following claim further restricts the range of their locations.

Claim 3. (1) $a_0 \in \{0, 1\}$, $a_4 \in \{m+3, m+4\}$; (2) $a_1 - a_0, a_4 - a_3 \in \{1, 2\}$; (3) if $a_4 = m + 3$, then $a_0 = 0$.

(1) Suppose $a_0 = 2$, then $a_1 = 3$, $a_3 = m + 3$ and $a_4 = m + 4$ by Claim 2. Since $a_1a_3 \in E(H)$, $a_2 = 5$ or $m + 1$ by taking $b_i = a_i$ $(i = 1, 2, 3)$ in Lemma 3.1(2), then $a_0a_2, a_2a_4 \in E(H)$, and thus a_0, a_1, a_2, a_3, a_4 form a 5-cycle, a contradiction. Similarly, $a_4 \in \{m+3, m+4\}$.

(2) By Claim 2, $a_1 - a_0 \in [1, 3]$. If $a_1 - a_0 = 3$, then $a_0 = 0$, $a_1 = 3$ and thus $a_4 = m+4$. Since $a_3 \in [m+1, m+3]$, we have $a_1a_3 \in E(H)$. Hence $a_2 = a_1 - 2 = 5$ or $a_2 = a_3 - 2 \in [m-1, m+1]$ by Lemma 3.1(2), and $a_2a_4 \in E(H)$. Since either $a_1a_2 \in E(H)$ or $a_2a_3 \in E(H)$, there is always a (a_3, a_4)-path and so we have $a_3a_4 \notin E(H)$, i.e., $a_3 = a_4 - 2 = m + 2$. Hence a_0, a_1, a_2 induce a triangle when $a_2 = m$ and a_0, a_1, a_3, a_2 induce a 4-cycle when $a_2 = 5$, a contradiction. Similarly, $a_4 - a_3 \in \{1, 2\}$.

(3) If $a_4 = m + 3$, then $a_1 \le 2$ by Claim 2. If $a_0 = 1$, then $a_1 = 2$ and $a_3 = m + 2$. Since $a_0a_1 \in E(H)$, $a_2 = 3$ or 4 by taking $b_i = a_{i-1}$ $(i = 1, 2, 3)$ in Lemma 3.1(3) and so a_2, a_3, a_4 induce a triangle, a contradiction. We conclude $a_0 = 0$.

Claim 4. There are at most five vertices receiving the color α in H.

Suppose, to the contrary, that the color α is used on six vertices $0 \le a_0 < a_1 < \cdots < a_5 \le m + 4$ in H. By Claim 2, it yields $a_5 - a_2 > m$, $a_4 - a_1 > m$ and $a_3 - a_0 > m$, and so $a_3 \in \{m+1, m+2\}$, $a_4 \in \{m+2, m+3\}$, $a_5 \in \{m+3, m+4\}$, $a_0 \in \{0, 1\}$, $a_1 \in \{1, 2\}$ and $a_2 \in \{2, 3\}$. Moreover, $a_2a_3 \in E(H)$ since $6 \le a_3 - a_2 \le m$. By Lemma 3.1(3), then $a_1 = 1$ (otherwise, if $a_1 = 2$, then $a_1a_2, a_1a_3 \in E(H)$ and a_1, a_2, a_3 induce a triangle) and $a_3 = m + 2$ (otherwise, if $a_3 = m + 1$, then $a_1a_3 \in E(H)$ and $a_2 = 3$, so

a_0, a_1, a_3, a_2 induce a 4-cycle). Hence, $a_0 = 0$, $a_4 = m+3$, $a_5 = m+4$ and $a_0a_1, a_4a_5, a_3a_4 \in E(H)$. We also see $a_2 = 2$ by taking $b_i = a_{i+1}$ $(i = 1, 2, 3)$ in Lemma 3.1(3). For the remaining $m-1$ vertices $3, 4, \ldots, m+1$ in H, there are $q-1$ colors in which each color β induces an F-type set V_{β_v} $(v \geq 3)$ plus one more color γ is used on three vertices $3 \leq h_1 < h_2 < h_3 \leq m+1$. Since $m+5, m+6, m+7 \notin V_\alpha \cup V_{\beta_v} (v \geq 3)$, we have $m+5, m+6, m+7 \in V_\gamma$, then $h_3 \leq (m+6)-(m+1) = 5$ by taking $b_1 = h_3$, $b_2 = m+5$, $b_3 = m+6$ in Lemma 3.1(3). Thus $3 \leq h_1 \leq h_3 - 2 \leq 3$, that is, $h_1 = 3$. As a result, each color β induces an F-type set V_{β_v} with $v \geq 4$, and then $m+8 \notin V_\alpha \cup V_{\beta_v}$. So $m+8 \in V_\gamma$, but $m+8$ induces a 4-cycle along with $m+5, m+6, m+7$, a contradiction.

Claim 5. Except α, any other color is used on exactly four vertices in H.

By Claim 4, each color is used on at most five vertices. To see this claim, we only need to show that there exists no other color, except α, used on five vertices in H.

Assume, to the contrary, that there exists a color $\alpha'(\neq \alpha)$ used on five vertices $0 \leq c_0 < c_1 < \cdots < c_4 \leq m+4$. By Claim 3, $a_0, c_0 \in \{0, 1\}$, $a_1, c_1 \in \{2, 3\}$, $a_3, c_3 \in \{m+1, m+2\}$ and $a_4, c_4 \in \{m+3, m+4\}$. Without loss of generality, assume that $a_0 = 0$, then $c_0 = 1$, $a_1 = 2$, $c_1 = 3$, $a_3 = m+1$, $c_3 = m+2$, $a_4 = m+3$ and $c_4 = m+4$ by Claim 3. Since $a_1a_3, c_1c_3 \in E(H)$, we have $a_2 \in \{4, m-1\}$ and $c_2 \in \{5, m\}$ by Lemma 3.1(2). Hence $R = [0, m+4] \setminus \{a_i, c_i \mid 0 \leq i \leq 4\} = [4, m] \setminus \{a_2, c_2\}$. By Claim 2, there is no other color used on five vertices in R. Thus there are $q-1$ colors in which each is used on a F-type set $V_{\beta_v} (v \geq 4)$ except a color γ is used on three vertices $4 \leq h_1 < h_2 < h_3 \leq m$ in R. Since there always exists an (a_2, a_3)-path and a (c_2, c_3)-path, we see $m+6, m+7, m+8 \notin V_\alpha$, $m+5, m+7, m+8 \notin V_{\alpha'}$, and $m+5, m+6, m+7, m+8 \notin V_{\beta_v} (v \geq 4)$ by Lemma 3.1(1) and Lemma 3.3. Hence $m+7, m+8 \in V_\gamma$ and $h_3 \leq 7$ by taking $b_1 = h_3$, $b_2 = m+7$, $b_3 = m+8$ in Lemma 3.1(3). It follows that $\{4, 5\} \cap \{h_1, h_2, h_3\} \neq \emptyset$, i.e., $a_2 \neq 4$ or $c_2 \neq 5$. Since $a_2 \neq 4$ implies $m+5 \notin V_\alpha$ (otherwise, if $m+5 \in V_\alpha$, then $a_1, a_3, m+5, a_2$ induce a 4-cycle), and $c_2 \neq 5$ and $a_2 = 4$ implies $m+6 \notin V_{\alpha'}$ (otherwise, $c_1, c_3, m+6, c_2$ induce a 4-cycle), we have $\{m+5, m+6\} \cap V_\gamma \neq \emptyset$. So there exists either an $(m+5, m+7)$-path or an $(m+6, m+7)$-path in $\langle V_\gamma \rangle$. Hence $h_3 \leq m+7-(m+1) = 6$ by Lemma 3.1(3), and then $h_1 = 4$, $h_2 = 5$, $m+5, m+6 \in V_\gamma$ and vertices $m+5, m+6, m+7, m+8$ induce a 4-cycle in $\langle V_\gamma \rangle$, a contradiction.

By Claim 3, if $a_4 = m+3$, we have $a_0 = 0$. Then, by Lemma 3.2, the subgraph H' induced by vertices $[-m-4, 0]$ also has a tree $(q+1)$-coloring. That is, Claims 1–2 and Claims 4–5 still hold in H'. Thus, if we can get a contradiction in H for $a_4 = m+4$, then there is a contradiction in H' for $a_0 = 0$ similarly. Therefore, we only need to consider the case of $a_4 = m+4$.

Let $\bar{a}_{ij} = A \setminus \{a_i, a_j\}$, where $\{a_i, a_j\} \subset A$, $a_i \neq a_j$ and $|A| = 3$. We can define $\bar{a}_i$ and $\bar{a}_{ijk}$ similarly.

In the following, we denote $[0, m+4] \setminus \{a_i, 0 \leq i \leq 4\}$ by R, and will derive a contradiction to $a_4 = m+4$. By Claim 3, $a_3 \in \{m+3, m+2\}$, thus there are only two cases to consider.

Case 1. $a_3 = m+3$.

Then $a_3a_4 \in E(H)$ and $a_2 \in \{2, 3, m+1, m+2\}$ by Lemma 3.1(3).

If $a_2 \in \{m+1, m+2\}$, then either $a_2a_3 \in E(H)$ or $a_2a_4 \in E(H)$. So there exists an (a_2, a_3)-path in $\langle V_\alpha \rangle$ and then $a_1 \leq 2$ by Lemma 3.1(3). Hence $R = \{\bar{a}_{01}\} \cup [3, m] \cup \{\bar{a}_2\}$, where $\bar{a}_{01} = \{0, 1, 2\} \setminus \{a_0, a_1\}$ and $\bar{a}_2 = \{m+1, m+2\} \setminus \{a_2\}$. Let γ color $\bar{a}_{01} = h_1 < h_2 < h_3 < h_4 \leq \bar{a}_2$, then any other color must induce an F-type set $V_{\beta_v} (v \geq 3)$ in R. By Lemma 3.3, $m+5, m+6 \notin V_\alpha \cup V_{\beta_v} (v \geq 3)$ (since $m+5$ induces a cycle along with a_4 and an (a_2, a_3)-path, and $m+6$ induces another cycle along with an (a_2, a_3)-path), $m+5, m+6 \in V_\gamma$. Thus $h_4 \leq 5$ by Lemma 3.1(3), but we always have $h_4 \geq \bar{a}_{01} + 6 \geq 6$ by Claim 1(3), a contradiction. Therefore $a_2 \in \{2, 3\}$ and $R = \{\bar{a}_{012}\} \cup [4, m+2]$, where $\bar{a}_{012} = \{0, 1, 2, 3\} \setminus \{a_0, a_1, a_2\}$. Let γ' color $\bar{a}_{012} = u_1 < u_2 < u_3 < u_4 \leq m+2$, then any other color must induce an F-type set in R. Since $m+7, m+8 \notin V_\alpha \cup V_{\beta_v} (v \geq 4)$, we have $m+7, m+8 \in V_{\gamma'}$. By Claim 1(3), if $\bar{a}_{012} \in \{0, 1\}$, then $u_4 \in \{m+1, m+2\}$; and if $\bar{a}_{012} \in \{2, 3\}$, then $u_4 = \bar{a}_{012} + 6 \in \{8, 9\}$. In either case, $u_4, m+7, m+8$ form a triangle, a contradiction.

Case 2. $a_3 = m+2$.

For $a_1 = 1$ (and so $a_0 = 0$), let H' be the subgraph induced by vertices $[-m-3, 1]$, then, by Lemma 3.2, we can obtain a contradiction in H' similar to the case $a_3 = m+3$ and $a_4 = m+4$ in H. Thus $a_1 \in \{2, 3\}$, $a_1a_3 \in E(H)$, and $a_2 \in \{a_1+2, m\}$ by Lemma 3.1(2). Moreover, $a_0a_2 \in E(H)$ and either $a_1a_2 \in E(H)$ or $a_2a_3 \in E(H)$. So there exists an (a_0, a_1)-path and thus $a_0a_1 \notin E(H)$, i.e., $a_1 = a_0 + 2$ and $a_2 \in \{4, 5, m\}$. Since $a_2a_4 \in E(H)$ and there exists an (a_3, a_4)-path in $\langle V_\alpha \rangle$, $m+5, m+7, m+8, m+9 \notin V_\alpha$ and $R = \{\bar{a}_0, \bar{a}_0+2\} \cup [4, m+1] \cup \{m+3\} \setminus \{a_2\}$, where $\bar{a}_0 = \{0, 1\} \setminus \{a_0\}$. Let γ color four vertices $\bar{a}_0 = h_1 < h_2 < h_3 < h_4 \leq m+3$ in R.

Subcase 2.1. $h_4 - h_1 \leq m$.

In this case, any color, except α, is used on an F-type set V_{β_v} which satisfies $v = \bar{a}_0$ or $v \geq 4$. If $a_2 = m$, then $m+5, m+6, m+7, m+8 \notin V_\alpha \cup V_{\beta_v} (v \geq 4)$, and thus $m+5, m+6, m+7, m+8$ belong to V_γ and induce a 4-cycle, a contradiction. If $a_2 \neq m$, then $a_2 \in \{4, 5\}$. Since $\bar{a}_0 + 4 \in \{4, 5\}$, we have $\{4, 5\} \subseteq V_\alpha \cup V_\gamma$. Then any other color β induces an F-type set V_{β_v} with $v \geq 6$. Since $m+5, m+8, m+9 \notin V_\alpha \cup V_{\beta_v} (v \geq 6)$, $m+5, m+8, m+9$ belong to V_γ and form a triangle, a contradiction.

Subcase 2.2. $h_4 - h_1 \geq m+1$.

If $h_4 = m+1$, then $\bar{a}_0 = 0$ and thus there exists a color, say γ', used on 2 and $m+3$ (otherwise, $m+1$ and $m+3$ receive the same color by Claim 1(3)). Let γ' color $2 = g_1 < g_2 < g_3 < g_4 = m+3$. By Claim 1(2), $h_2 = m-3$, $h_3 = m-1$, $g_2 = 4$ and $g_3 = 6$. Since $m+5, m+8, m+9 \notin V_\alpha \cup V_{\beta_v} (v \geq 5)$ and $m+5 \notin V_\gamma$, we have $m+5 \in V_{\gamma'}$, $m+8 \in V_\gamma$, and then $m+9 \in V_\gamma \cup V_{\gamma'}$ but it induces a triangle along with vertices $h_4, m+8$, or a 4-cycle along with vertices $g_4, g_3, m+5$, a contradiction.

Thus $h_4 = m+3$, and $h_2 = h_1 + 2 = \bar{a}_0 + 2$ or $h_3 = m+1$ by Claim 1(1). If $h_1 = \bar{a}_0 + 2$, then, for any other color β, the F-type set V_{β_v} satisfies $v \geq 4$. Since $m+7, m+8 \notin V_\alpha \cup V_{\beta_v} (v \geq 4)$, $m+7, m+8$ belong to V_γ and form a triangle with $m+3$, a contradiction. If $h_3 = m+1$ and $h_2 > \bar{a}_0 + 2$, then, for any other color β, the F-type set V_{β_v} has $v = \bar{a}_0 + 2$ or $v \geq 4$. Let γ'

color $\bar{a}_0 + 2$. As there exists an (h_3, h_4)-path when $h_2 \neq m-1$ and an (h_2, h_4)-path when $h_2 = m-1$, we have $m+7 \notin V_\gamma$. Note that $m+5, m+7, m+8 \notin V_\alpha \cup V_{\beta_v} (v \geq 4)$ and $m+5 \notin V_{\gamma'}$, we have $m+5 \in V_\gamma$ and $m+7 \in V_{\gamma'}$, and then either $m+8$ belongs to V_γ and induces a triangle along with vertices $h_3 = m+1$ and $m+5$, or $m+8$ belongs to V'_γ and induces a triangle along with vertices $m+7$ and $\bar{a}_0+8$, a contradiction again.

After all, we have shown that $va(G(D_{m,2})) \geq \lceil \frac{m+1}{4} \rceil + 1$ for $m = 4q \geq 8$.

Next, for $m = 4q + j > 8$ with $0 < j \leq 3$, we see $va(G(D_{m,2})) \geq va(G(D_{4q,2})) \geq \lceil \frac{4q+1}{4} \rceil + 1 = \lceil \frac{m+1}{4} \rceil + 1$.

For $m = 6$, let G_1 be the subgraph induced by vertices $[0, 8]$. If $va(G(D_{6,2})) = 2$, then $G(D_{6,2})$ has a tree 2-coloring f_1 which is also a tree coloring of G_1. Note that $|V(G_1)| = 9$. There are at least five vertices, say $0 \leq a_0 < a_1 < \cdots < a_4 \leq 8$, receiving the same color α. Then Claims 1–2 hold. So $a_0 = 0$, $a_1 = 1$, $a_3 = 7$ and $a_4 = 8$. If $a_2 > 2$, then $a_0a_1, a_0a_2 \in E(G_1)$, so $a_1a_2 \notin E(G_1)$, i.e., $a_2 = 3$. Hence, a_2, a_3, a_4 induce a triangle, a contradiction. If $a_2 = 2$, then a_2, a_3, a_4 induce a triangle, too. Therefore, $va(G(D_{6,2})) \geq 3$, and then $va(G(D_{7,2})) \geq va(G(D_{6,2})) \geq 3 = \lceil \frac{7+1}{4} \rceil + 1$.

Therefore, the lower bound is confirmed. □

Now we present an algorithm for finding a tree coloring of the integer distance graph $G(D_{m,2})$.

If $m \leq 5$, then assign $r = x \pmod 2 \in [0, 1]$ to each vertex x and obtain a tree coloring of $G(D_{m,2})$. For $m \geq 6$, let $m = 8l + j \geq 6$ with $0 < j \leq 8$.

Algorithm. $A(m, 2)$. If $0 < j \leq 3$, then go to A1; if $4 \leq j \leq 6$, then go to A2; if $7 \leq j \leq 8$, then go to A3. Repeat the process until each vertex is colored.

A1: For any vertex x, if x can be written as $x = 8t+2s+r$ for $0 \leq t \leq l$, $s \in [0, 3]$ and $r \in [0, 1]$, then we define $f(x) = 2t+r$; otherwise, x can be written as $x = 8(l+1)n + x'$ for some $0 \leq x' < 8(l+1)$ and $n \in \mathbb{Z}$, and then we define $f(x) = f(x')$.

A2: Let $u = 8(l+1)+3$. For any vertex x, if x can be written as $x = 8t+2s+r$ for $0 \leq t \leq l$, $s \in [0, 3]$ and $r \in [0, 1]$, then we define $f(x) = 2t+r$; if $x \in [u-3, u-1]$, then we define $f(x) = 2(l+1)$; if $x \notin [0, u-1]$, then x can be written as $x = un + x'$ for some $0 \leq x' \leq u-1$ and $n \in \mathbb{Z}$, and we define $f(x) = f(x')$.

A3: For any vertex x, if x can be written as $x = 8t+2s+r$ for $0 \leq t \leq l+1$, $s \in [0, 3]$ and $r \in [0, 1]$, then we define $f(x) = 2t+r$. Otherwise, then x can be expressed as $x = 8(l+2)n + x'$ for some $0 \leq x' < 8(l+2)$ and $n \in \mathbb{Z}$, and we define $f(x) = f(x')$.

4. Vertex arboricity of $G(D_{m,k})$

In the last section, we investigate vertex arboricity of $G(D_{m,k})$ for $k \geq 3$.

Suppose $m \leq k + \lfloor \frac{k}{2} \rfloor - 1$. Since vertices $[0, k-1]$ induce a complete subgraph of order k, $va(G(D_{m,k})) \geq \lceil \frac{k}{2} \rceil$. We define a tree coloring f: $f(kl+i) \equiv i \pmod{\lceil \frac{k}{2} \rceil}$ for $l \in \mathbb{Z}$ and $0 \leq i < k$, that is, for every $0 \leq i \leq \lfloor \frac{k}{2} \rfloor - 1$, the vertices in $V_i = \{\ldots, i, \lceil \frac{k}{2} \rceil + i, k+i, k+\lceil \frac{k}{2} \rceil + i, 2k+i, \cdots\}$ receive a color i. Obviously V_i induces a forest, as $2k+i-(\lceil \frac{k}{2} \rceil + i) = k + \lfloor \frac{k}{2} \rfloor > m$. If k is odd, then $V_{(k-1)/2} = \{\ldots, (k-1)/2, k+(k-1)/2, 2k+(k-1)/2, \cdots\}$ is an independent set. So f is a tree $\lceil \frac{k}{2} \rceil$-coloring, i.e., $va(G(D_{m,k})) \leq \lceil \frac{k}{2} \rceil$. Therefore, $va(G(D_{m,k})) = \lceil \frac{k}{2} \rceil$.

Suppose $k + \lfloor \frac{k}{2} \rfloor \leq m \leq 2k-1$. By Lemma 2.1, $va(G(D_{m,k})) \leq \lceil \frac{m}{2} \rceil$. Let H be a subgraph of $G(D_{m,k})$ induced by vertices $[0, m]$, then H is a complete k-partite graph $K(2, \ldots, 2, 1, \ldots, 1)$ with k-partite $X_0 = \{0, k\}, X_1 = \{1, k+1\}, \ldots, X_{m-k} = \{m-k, m\}, X_{m-k+1} = \{m-k+1\}, \ldots, X_{k-1} = \{k-1\}$. It is obvious that any four vertices of H induce a cycle, and any three vertices, which are contained in three partite respectively, induce a triangle. So $va(H) = 2k-m-1+\lceil 2\frac{m-k+1-(2k-1-m)}{3} \rceil = \lceil \frac{m+1}{3} \rceil$ since $0 \leq 2k-m-1 = (k-1)-(m-k) \leq \lceil \frac{k}{2} \rceil \leq (m-k)+1 \leq k$. Therefore, $va(G(D_{m,k})) \geq va(H) = \lceil \frac{m+1}{3} \rceil$ for $2k-1 \geq m \geq k + \lfloor \frac{k}{2} \rfloor$.

If $2k \leq m < 3k$, then $va(G(D_{m,k})) \leq k$ by Lemma 2.1. Let $X'_0 = \{0, k, 2k\}, X'_1 = \{1, k+1, 2k+1\}, \ldots, X'_{m-2k} = \{m-2k, m-k, m\}, X'_{m-2k+1} = \{m-2k+1, m-k+1\}, \ldots, X'_{k-1} = \{k-1, 2k-1\}$, then $X'_0 \cup X'_1 \cup \cdots \cup X'_{k-1} = [0, m]$ induces a supergraph H' of a complete k-partite graph $K(3, 3, \ldots, 3, 2, \ldots, 2)$. It is clear that any four vertices of H' induce a cycle and each X'_i $(0 \leq i \leq m-2k)$ requires a color. Hence, $va(H') = (m-2k)+1+\lceil 2\frac{k-1-(m-2k)}{3} \rceil = \lceil \frac{m+1}{3} \rceil$ and then $va(G(D_{m,k})) \geq \lceil \frac{m+1}{3} \rceil$. That is, $\lceil \frac{m+1}{3} \rceil \leq va(G(D_{m,k})) \leq k$ or $va(G(D_{m,k})) = k$ for $3k-3 \leq m < 3k$.

To summarize the above discussion, we have the following theorem:

Theorem 4.1. *For $k \leq m < 3k$, the vertex arboricity of $G(D_{m,k})$ is*

(1) $va(G(D_{m,k})) = \lceil \frac{k}{2} \rceil$ *for* $m \leq k + \lfloor \frac{k}{2} \rfloor - 1$;
(2) $\lceil \frac{m+1}{3} \rceil \leq va(G(D_{m,k})) \leq \lceil \frac{m}{2} \rceil$ *for* $k + \lfloor \frac{k}{2} \rfloor \leq m \leq 2k-1$;
(3) $\lceil \frac{m+1}{3} \rceil \leq va(G(D_{m,k})) \leq k$ *for* $2k \leq m < 3k$. *In particular,* $va(G(D_{m,k})) = k$ *for* $3k-3 \leq m < 3k$.

Next, we consider $m \geq 3k$ and will need the following from [1] as a lemma.

Lemma 4.1. *Suppose $m \geq 2k$. Write $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. Then*

$$\chi(G(D_{m,k})) = \begin{cases} \dfrac{m+k+1}{2} & \text{if } r > s; \\ \left\lceil \dfrac{m+k+2}{2} \right\rceil & \text{otherwise}. \end{cases}$$

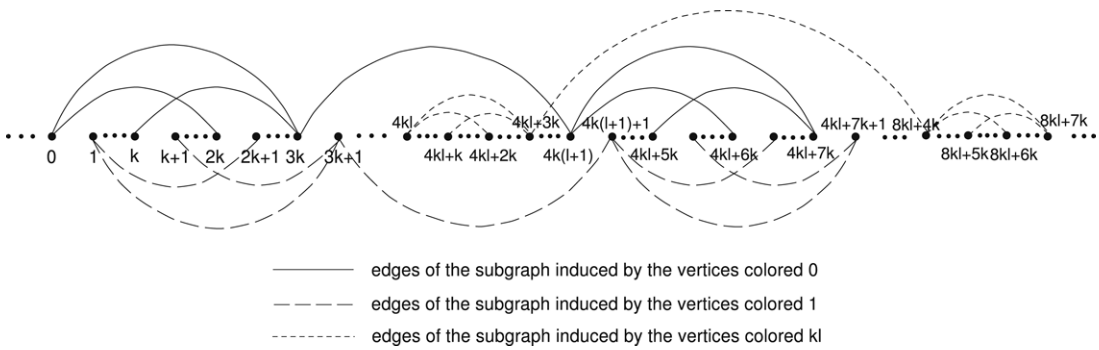

Fig. 4. A tree coloring for $m = 4kl + j \geq 3k \geq 9$, $k \leq j < 2k$ and $0 \leq n < 4(l+1)$ in $G(D_{m,k})$ $(k \geq 3)$.

Theorem 4.2. *Let $m = 4kl + j \geq 3k \geq 9$ with $0 \leq j < 4k$, then $\lceil \frac{m+k+1}{4} \rceil \leq va(G(D_{m,k})) \leq k\lceil \frac{m+2k+1}{4k} \rceil$. Moreover,*

$$va(G(D_{m,k})) \leq \begin{cases} k\left(\left\lfloor \dfrac{m}{4k} \right\rfloor + 1\right), & \text{for } 0 \leq j < 2k, \\ \left\lceil \dfrac{m}{4k} \right\rceil k + \left\lceil \dfrac{j-2k+1}{2} \right\rceil, & \text{for } 2k \leq j < 3k, \\ \left\lceil \dfrac{m}{4k} \right\rceil k + \left\lceil \dfrac{k}{2} \right\rceil, & \text{for } 3k \leq j < 3k + \left\lfloor \dfrac{k}{2} \right\rfloor - 1, \\ \left(\left\lceil \dfrac{m}{4k} \right\rceil + 1\right) k, & \text{for } 3k + \left\lfloor \dfrac{k}{2} \right\rfloor - 1 \leq j < 4k. \end{cases}$$

Proof. To show the upper bound, we construct a tree coloring of $G(D_{m,k})$ periodically as follows.

For $0 \leq j < 2k$ and $0 \leq n < 4k(l+1)$, let $f_1(x) = i + kt$ for $x - (i + 4kt) \in \{0, k, 2k, 3k\}$, $0 \leq i < k$ and $0 \leq t \leq l$; and $f_1(x + 4ks(l+1)) = f_1(x)$ for any $s \in \mathbb{Z}$. By Lemma 2.2, each of $V_{t,i} = \{4k(l+1)s + 4kt + i + kr \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ induces a forest and thus f_1 is a tree coloring (see Fig. 4). So $va(G(D_{m,k})) \leq (l+1)k = (\lfloor \frac{m}{4k} \rfloor + 1)k = k\lceil \frac{m+2k+1}{4k} \rceil$.

If $2k \leq j < 3k$, let

$$f_2(x) = \begin{cases} i + kt & \text{for } x - (4kt + i) \in \{0, k, 2k, 3k\},\ 0 \leq i < k, 0 \leq t \leq l, \\ k(l+1) + \left\lfloor \dfrac{n - 4k(l+1)}{2} \right\rfloor & \text{for } 4k(l+1) \leq x \leq m + 2k, \end{cases}$$

and other vertices be colored periodically. By Lemma 2.2, all vertex subsets $V'_{t,i} = \{(m+2k+1)s+4kt+i+kr \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ and $V'_{k(l+1)+u} = \{(m + 2k + 1)s + 4k(l + 1) + 2u + r \mid s \in \mathbb{Z},\ r \in [0, 1]\}$ (where $0 \leq u \leq \lceil \frac{j-2k+1}{2} \rceil - 1$) induce forests and then f_2 is a tree coloring. So $va(G(D_{m,k})) \leq \lceil \frac{m}{4k} \rceil k + \lceil \frac{m+2k-4k(l+1)+1}{2} \rceil = \lceil \frac{m}{4k} \rceil k + \lceil \frac{j-2k+1}{2} \rceil \leq k\lceil \frac{m+2k+1}{4k} \rceil$.

If $3k \leq j < 3k + \lfloor \frac{k}{2} \rfloor$, for $0 \leq x \leq m + 2k$, let

$$f_3(x) = \begin{cases} i + kt & \text{for } x - (4kt + i) \in \{0, k, 2k, 3k\},\ 0 \leq i < k, 0 \leq t \leq l, \\ k(l+1) + i & \text{for } x - i - 4k(l+1) = 0, \left\lceil \dfrac{k}{2} \right\rceil, k,\ 0 \leq i < \left\lceil \dfrac{k}{2} \right\rceil, \end{cases}$$

and other vertices be colored periodically. By Lemma 2.2, all vertex subsets $\overline{V}_{t,i} = \{(4k(l+1) + k + \lceil \frac{k}{2} \rceil)s + 4kt + i + kr \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ and $\overline{V}_{k(l+1)+u} = \{(4k(l+1) + k + \lceil \frac{k}{2} \rceil)s + 4k(l+1) + u + r \mid s \in \mathbb{Z},\ r \in \{0, \lceil \frac{k}{2} \rceil, k\}\}$ (where $0 \leq u < \lceil \frac{k}{2} \rceil$) induce forests and thus f_3 is a tree coloring. So $va(G(D_{m,k})) \leq \lceil \frac{m}{4k} \rceil k + \lceil \frac{k}{2} \rceil \leq k\lceil \frac{m+2k+1}{4k} \rceil$.

If $3k + \lfloor \frac{k}{2} \rfloor \leq j < 4k$, for $0 \leq x < 4k(l+2)$, let $f_4(x) = i + kt$ for $x - (i + 4kt) \in \{0, k, 2k, 3k\}$, $0 \leq i < k$ and $0 \leq t \leq l+1$; and $f_4(x + 4ks(l+2)) = f_4(x)$ for each $s \in \mathbb{Z}$. By Lemma 2.2, each vertex subset $\widehat{V}_{t,i} = \{4k(l+2)s + 4kt + i + kr \mid s \in \mathbb{Z},\ r \in [0, 3]\}$ induces a forest and then f_4 is a tree coloring. So $va(G(D_{m,k})) \leq (l+2)k = (\lceil \frac{m}{4k} \rceil + 1)k = k\lceil \frac{m+2k+1}{4k} \rceil$.

Next, we consider the lower bound. Let $n = \lceil \frac{m+k+1}{4} \rceil - 1 = \lceil \frac{m+k-3}{4} \rceil$. Assume, to the contrary, that $va(G(D_{m,k})) \leq n$. Then $\chi(G(D_{m,k})) \leq 2n < \lceil \frac{m+k+1}{2} \rceil$, a contradiction to Lemma 4.1.

Therefore, $va(G(D_{m,k})) \geq \lceil \frac{m+k+1}{4} \rceil$. □

We present the following remarks as a conclusion of this paper.

Remarks. 1. In Theorem 3.1, the only undetermined value is $va(G(D_{8q+7,2}))$. Between the two possible values, we believe that the correct value should be $\lceil \frac{m}{4} \rceil + 2$.

2. Let $D_{m,k,s} = [1, m] \setminus \{k, 2k, \dots, sk\}$. Some evidence suggests:

$$va(G(D_{m,1,s})) = \left\lceil \frac{m+s+2}{s+3} \right\rceil$$

for any positive integer s.

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