Commutative Rings and Modules

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Theorem

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence of modules.Then* B *satisfies accending chain condition on submodules (resp.descending) iff* A *and* C *satisfy it.*

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Let f_i be the restriction of f on A_i and g_i be the restriction of g on B_i

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But $f_i(A_i) = f(A_i) = f(A) \cap B_i \subseteq B_i$ Therefore $f_i : A_i \to B_i$

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But $f_i(A_i) = f(A_i) = f(A) \cap B_i \subseteq B_i$ Therefore $f_i : A_i \to B_i$

Similarly $g_i : B_i \to C$ and $g_i(B_i) = g(B_i) = C_i$. Therefore $g_i : B_i \to C_i$

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Claim: $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ is a short exact sequence

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To prove $Im\ f_i = Ker\ g_i\ i = 1, 2, \cdots$

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To prove $Im\ f_i = Ker\ g_i\ i = 1, 2, \cdots$

$$y \in Im\ f_i \Leftrightarrow y = f_i(x) \text{ for some } x \in A_i$$

$$\Leftrightarrow y \in Ker\ g_i$$

Therefore $Im\ f_i = Ker\ g_i$

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Let $n = \max\{n_1, n_2\}$, $A_i = A_n$ and $C_i = C_n\ \forall i \geq n$

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For each $i \geq n$, there is a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \longrightarrow & 0 \end{array}$$

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By Short five lemma β is an isomorphism.

Short five lemma

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$\implies$ β is the identity map and $B_n = B_i \; \forall \; i \geq n$

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Where α and γ are the identity maps,[since $A_n = A_i$ and $C_n = C_i$]

β be the inclusion map [since $B_n \subseteq B_i$]

α , β , γ are homomorphism of modules and α , γ are isomorpisms.

By Short five lemma β is an isomorphism. Short five lemma

$\implies \beta$ is the identity map and $B_n = B_i \ \forall\ i \geq n$

Hence B satisfies ACC.

Corollary

If A is a submodule of a module B ,then B satisfies the ascending [resp.descending] chain conditions iff A and B/A satisfy it.

corollary 1.6

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Conclusion

Corollary

If $A_1, A_2, \cdots, A_n$ are submodules,then the direct sum$A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies the ascending [resp.descending] chain condition on submodules iff each A_i satisfies it.

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corollary 1.7

claim: $Im\ i_1 = Ker\ \pi_2$

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Thus the result is true for $n = 2$.By induction it is true for any n

start

Lemma (Short five lemma)

Let R be a Ring and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

a commutative diagram of R-module homomorphisms such that each row is a short exact sequence. Then

(i) α, γ *monomorphisms* $\Longrightarrow$ β *is a monomorphism.*

(ii) α, γ *epimorphisms* $\Longrightarrow$ β *is an epiorphism.*

(iii) α, γ *isomorphisms* $\Longrightarrow$ β *is an isomorphism.*

Return

Result

If a module satisfies ACC ,then every submodule satisfies it

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If a module satisfies ACC ,then every submodule satisfies it

Converse need not be true

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If a module satisfies ACC ,then every submodule satisfies it

Converse need not be true

If the submodule and the quotient module are satisfying ACC ,then the module satisfies it

Return

Exact Sequences

Definition

A finite sequence of module homomorphisms ,

$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$, is exact provided $Im\ f_i = Ker\ f_{i+1}$, for $i = 1, 2, \cdots, n$

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Note

In the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, f is a monomorphism and g is an epimorphism

Return