Numerical Methods for Singularly Perturbed Boundary Value Problems for Higher Order Ordinary Differential Equations

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Introduction to Singular Perturbation Problems(SPPs)

- The birth of the singular perturbations introduced by Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904 and it was reported in the proceedings of the conference .
- Many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems are described by differential equations involving large or small parameters.

Let P_ε denote the original problem and u_ε be its solution.
Let P_0 denote the reduced problem of P_ε (setting $\varepsilon = 0$ in P_ε) and u_0 be its solution. Then the problem P_ε is called a **Singularly Perturbed Problem (SPP)** if and only if u_ε does not converge uniformly to u_0 in the entire domain of the definition of the problem. Otherwise the problem is called **Regularly Perturbed Problem (RPP).**

Example 1.1 (Singular Perturbation Problem)

$$P_\varepsilon : \quad \begin{cases} \varepsilon u_\varepsilon'(x) = -u_\varepsilon(x), \ x \in (0,1], \\ u_\varepsilon(0) = 1, \quad 0 < \varepsilon \ll 1. \end{cases}$$

$$P_0 : \quad u_0(x) = 0, \ x \in [0,1],$$

The exact solution of P_ε is given by

$$u_\varepsilon(x) = e^{-x/\varepsilon}.$$

Note that,

$$\lim_{\varepsilon \to 0} \lim_{x \to 0} u_\varepsilon(x) = 1 \neq 0 = \lim_{x \to 0} \lim_{\varepsilon \to 0} u_\varepsilon(x).$$

That is, $u_\varepsilon(x)$ does not converge uniformly to the reduced problem solution on $[0, 1]$. The solution changes very rapidly near the neighborhood of $x = 0$. This neighborhood is called a **boundary layer**.

- A singular perturbation problem is said to be of convection-diffusion type, if the order of the differential equation is reduced by one when the perturbation parameter is set equal to zero.
- If the order reduces by two, it is known as a reaction-diffusion type problem.

- Often these mathematical problems are extremely difficult (or even impossible) to solve exactly and in these circumstances, approximate solutions are necessary. One can obtain an approximate solution through the use of perturbation methods. The basic idea of these methods is to begin with the solution of a simpler problem (as a first approximation) and then to obtain systematically better and better approximations.
- Typically these problems arise in various fields of applied mathematics such as fluid dynamics, elasticity, quantum mechanics, electrical networks, chemical reactor-theory, gas porous electrodes theory, aerodynamics, plasma dynamics, oceanography, diffraction theory, reaction-diffusion processes and many other areas.

Characteristics of Singular Perturbation Problems

- The solution of SPPs have a multiscale character(non-uniform behaviour), that is, there are thin layer(s)(Boundary layer region) where the solution varies rapidly while away from the layer(s) (Outer Region) the solution behave regularly and varies slowly.

Non- Classical Numerical Methods for SPPs

In general, classical numerical methods like Euler method, Runge Kutta methods, finite difference methods, etc cannot be applied to these SPPs. In the literature various non-classical methods are available.

(i) Variable Mesh size Method (VMM) (ii)Boundary Value Technique (BVT)

(iii) Initial Value Technique (IVT) (iv)Fitted Operator Method (FOM)

(v) Fitted Mesh Method (FMM) (vi)Booster Method (BM)

(vii) Schwarz Iterative Method (SIM) (viii)Shooting Method (SM)

(ix) Spline Approximation Method (SAM) (x)Finite Element Method (FEM)

(xii) Collocation Method.

Computational method

Motivated by the works of [19, 20], we suggested a new computational method which makes use of the zero order asymptotic expansion approximation, BVT and shooting method to obtain a numerical solution for the derivative of SPBVPs for third order ODEs of convection-diffusion type of the form:

$$\varepsilon y'''(x) + a(x)y''(x) - b(x)y'(x) - c(x)y(x) = f(x),\ x \in \Omega, \quad (1)$$

$$y(0) = p, \quad -y''(0) = q, \quad y'(1) - y''(1) = r. \quad (2)$$

where $y \in C^{(3)}(\Omega) \cap C^{(2)}(\bar{\Omega})$, $0 < \varepsilon \ll 1, a(x), b(x), c(x)$ and $f(x)$are sufficiently smooth functions satisfying the following conditions .

$$
\begin{aligned}
a(x) \geq \alpha, \alpha > 0, \\
b(x) > 0, \\
0 \geq c(x) \geq -\gamma, \gamma > 0, \\
\alpha > \gamma.
\end{aligned}
$$

The SPBVPs (1)-(2) can be transformed into an equivalent weakly coupled system of the form:

$$\begin{cases} P_1\bar{y}(x) \equiv -y_1'(x) + y_2(x) = 0,\ x \in \Omega^0, \\ P_2\bar{y}(x) \equiv \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) \\ -c(x)y_1(x) = f(x),\ x \in \Omega, \\ y_1(0) = p, \quad -y_2'(0) = q, \quad y_2(1) - y_2'(1) = r, \end{cases} \tag{3}$$

with $\Omega = (0,1), \quad \Omega^0 = (0,1], \quad$ where $\bar{y} = (y_1, y_2)^T,$ the functions $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the same conditions given above.

Analytical Results

Theorem 3.1

(Maximum Principle). *Consider the BVPs (3). Let* $y_1(0) \geq 0$, $y_2'(0) \leq 0$ *and* $y_2(1) - y_2'(1) \geq 0$. *Then* $P_1\bar{y}(x) \leq 0,$ *for* $x \in \Omega^0$ *and* $P_2\bar{y}(x) \leq 0,$ *for* $x \in \Omega$, *implies that* $\bar{y}(x) \geq 0, \forall x \in \bar{\Omega}$.

Lemma 3.2

(Stability Result). *If* $\bar{y}(x)$ *is the solution of the BVPs (3) then*

$$||\bar{y}(x)|| \leq C \max\{|y_1(0)|, |y_2'(0)|, |y_2(1) - y_2'(1)|, \max_{x\in\Omega^0}|P_1\bar{y}(x)|,$$

$$\max_{x\in\Omega}|P_2\bar{y}(x)|\}, \forall x \in \bar{\Omega}.$$

Asymptotic Expansion Approximation

We look for an asymptotic expansion solution of the BVPs (3) in the form

$$\bar{y}(x,\varepsilon) = (\bar{u}_0(x) + \bar{v}_0(x)) + \varepsilon(\bar{u}_1(x) + \bar{v}_1(x)) + \cdots$$

By the method of stretching variable [1] one can obtain a zero order asymptotic approximation as $\bar{y}_{as}(x) = \bar{u}_0(x) + \bar{v}_0(x)$, where $\bar{u}_0(x) = (u_{01}(x), u_{02}(x))^T$ is the solution of the reduced problem of the BVPs (3) given by

$$\begin{cases} -u'_{0_1}(x) + u_{0_2}(x) = 0, \\ a(x)u'_{0_2}(x) - b(x)u_{0_2}(x) - c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = p, \quad u_{0_2}(1) - u'_{0_2}(1) = r, \end{cases} \tag{4}$$

and $\bar{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is a layer correction term satisfies

$$\begin{cases} -v'_{0_1}(x) + v_{0_2}(x) = 0, \\ \varepsilon v''_{0_2}(x) + a(0)v'_{0_2}(x) = 0, \\ v_{0_1}(0) = -(\varepsilon/a(0))v_{0_2}(0), \\ v_{0_2}(0) = (q + u'_{0_2}(0)),\ v_{0_2}(1) = \exp(-(a(0)/\varepsilon))v_{0_2}(0) \end{cases} \tag{5}$$

and this $\bar{v}_0(x)$ given by

$$\begin{cases} v_{0_1}(x) = -(\varepsilon/a(0))(q + u'_{0_2}(0))\exp(-(a(0)/\varepsilon))(x), \\ v_{0_2}(x) = (q + u'_{0_2}(0))\exp(-(a(0)/\varepsilon))(x). \end{cases} \tag{6}$$

The following theorem gives the bound for the difference between the solution of the BVPs (3) and its zero order asymptotic expansion approximation.

Theorem 3.3

The zero order asymptotic approximation $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x)$ *of the solution* $\bar{y}(x)$ *of the BVPs (3) defined by (4)-(6) satisfies the inequality*

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \leq C\varepsilon, \quad \forall x \in \bar{\Omega}.$$

Corollary 3.4

If $y_1(x)$ *is the solution of the BVPs (3) and* $u_{0_1}(x)$ *is the solution of the problem (4) then* $|y_1(x) - u_{0_1}(x)| \leq C\varepsilon, \; \forall x \in \bar{\Omega}$.

Some analytical and numerical results for second order SPBVPs

Consider the auxiliary second order SPBVPs

$$Ly_2^\star(x) \equiv \varepsilon y_2^{\star''}(x) + a(x)y_2^{\star'}(x) - b(x)y_2^\star(x) = f(x) + c(x)u_{0_1}(x),\ x \in \Omega, \quad (7)$$

$$B_0 y_2^\star(0) \equiv -y_2^{\star'}(0) = q, \quad B_1 y_2^\star(1) \equiv y_2^\star(1) - y_2^{\star'}(1) = r, \quad (8)$$

where $u_{0_1}(x)$ is defined as in (4), $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth and $a(x) \geq \alpha$ and $b(x) > 0$, $0 \geq c(x) \geq -\gamma, \gamma > 0$.

Analytical Results

Theorem 3.5

(Maximum Principle). *Consider the BVPs (7)-(8). Let* $y_2^\star(x)$ *be a smooth function satisfying* $B_0 y_2^\star(0) \geq 0$, $B_1 y_2^\star(1) \geq 0$ *and* $L y_2^\star(x) \leq 0$ *for* $x \in \Omega$*. Then,* $y_2^\star(x) \geq 0, \forall x \in \bar{\Omega}$.

Lemma 3.6

(Stability Result). *If $y_2^\star(x)$ is the solution of the BVPs (7)-(8) then*

$$|y_2^\star(x)| \leq C \max\{|B_0 y_2^\star(0)|, |B_1 y_2^\star(1)|, \max_{x\in\Omega}|Ly_2^\star(x)|\}, \forall x \in \bar{\Omega}.$$

Theorem 3.7

If $\bar{y}(x)$ and $y_2^\star(x)$ are solutions of the BVPs (3) and (7)-(8) respectively, then $|y_2(x) - y_2^\star(x)| \leq C\varepsilon, \ \forall x \in \bar{\Omega}$.

Description of the method

Step 1: An asymptotic approximation is derived for the solution of (3) which is given by (4)-(5).
Step 2: The first component of the solution $\bar{y}(x)$ of the BVPs (3), namely y_1 is approximated by the first component of the solution of the reduced problem namely u_{0_1} given by (4). Then replacing y_1 appearing in the second equation of (3) by u_{0_1} and taking the same boundary values, one gets the auxiliary SPBVP (7)-(8). The solution of this problem is taken as an approximation to y_2 which is the second equation of (3) which has to be solved.
Step 3: In order to solve the auxiliary second order problem (7)-(8) numerically, we divide the interval $[0, 1]$ into two subintervals $[0, \tau]$ and $[\tau, 1]$ called inner and outer region respectively,
where $\tau = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha} \ln N\}$.

Then, from the BVPs (7)-(8) two problems namely inner region problem and outer region problem are derived. To find the boundary condition at $x = \tau$, a zero order asymptotic expansion is used.
The inner region problem for (7)-(8) is given by

$$\begin{cases} \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), x \in (0,\tau), \\ -\tilde{y}_2'(0) = q,\ y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}. \end{cases} \tag{9}$$

The outer region problem for (7)-(8) is given by

$$\begin{cases} \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), x \in (\tau,1), \\ y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}, y_2(1) - y_2'(1) = r, \end{cases} \tag{10}$$

Numerical Schemes

Inner region problem:

Step 4: The inner region problem is solved by the Shooting method. Here,Shooting method in the sense that the BVPs (9) is replaced by the IVPs (11) in the interval $[0, \tau]$.

Consider the following IVPs:

$$\begin{cases} \varepsilon \tilde{y}_2''(x) + a(x)\tilde{y}_2'(x) - b(x)\tilde{y}_2(x) = f(x) + c(x)u_{0_1}(x),\ x \in (0, \tau], \\ \tilde{y}_2(0) = \bar{q} = u_{0_2}(0) + v_{0_2}(0), -\tilde{y}_2'(0) = q. \end{cases} \tag{11}$$

This IVP is equivalent to the system

$$P^*\bar{y}^* = \begin{cases} P_1^*\bar{y}^* = -y_1^{*\prime}(x) + y_2^*(x) = 0, \\ P_2^*\bar{y}^* = \varepsilon y_2^{*\prime}(x) + a(x)y_2^*(x) - b(x)y_1^*(x) = f^*(x),\ x \in (0, \tau], \\ y_1^*(0) = \bar{q}, -y_2^*(0) = q, \end{cases} \tag{12}$$

Applying the Euler's Finite Difference scheme on (12),we get

$$\begin{cases} P_1^{*N/2}\bar{y}_i^* = -D^- y_{1,i}^* + y_{2,i}^* = 0, \\ P_2^{*N/2}\bar{y}_i^* = \varepsilon D^- y_{2,i}^* + a(x_i)y_{2,i}^* - b(x_i)y_{1,i}^* = f^*(x_i),\ 1 \leq i \leq N/2, \\ y_{1,0}^* = \bar{q}, -y_{2,0}^* = q, \end{cases} \tag{13}$$

where, $D^- y_{j,i}^* = (y_{j,i}^* - y_{j,i-1}^*)/h_1,\ h_1 = \dfrac{2\tau}{N},\ x_i = x_{i-1} + ih_1,$

$j = 1, 2$. Here, τ is the transition parameter given by $\tau = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha} \ln N\}$. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Outer region problem

Step 5: The outer region problem given in (10) is solved by the standard FD scheme. Appling the CFD scheme on (10), we get

$$\begin{cases} L^{N/2}y_{2,i} := \varepsilon\delta^2 y_{2,i} + a(x_i)D^+y_{2,i} - b(x_i)y_{2,i} \\ = f(x_i) + c(x_i)u_{0_1}(x_i), 1 \leq i \leq N/2-1, \\ B_0^{N/2}y_{2,0} = \bar{r}, B_1^{N/2}y_{2,N} = y_{2,N/2} - (y_{2,N/2} - y_{2,N/2-1})/h_2 = r, \end{cases} \tag{14}$$

where, $D^+y_{2,i} = (y_{2,i+1} - y_{2,i})/h_2, \quad \delta^2 y_{2,i} = (y_{2,i+1} - 2y_{2,i} + y_{2,i-1})/h_2^2,$

$$x_i = x_{i-1} + ih_2, \quad \text{and} \quad h_2 = \frac{2(1-\tau)}{N}.$$

Here, τ is defined as mentioned above. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Step 6:

After solving both the inner region and outer region problems, we combine their solutions to obtain an approximate solution y_2 for the derivative of the original problem (1)-(2) over the interval $\bar{\Omega}$.

Error Estimates

Inner region problem

Theorem 3.8

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be respectively, the solutions of (12) and (13). Then h

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1} \ln N \quad \text{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}$$

Theorem 3.9

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ *and* $\bar{y}^{*1} = (y_1^{*1}, y_2^{*1})^T$ *be respectively, the solutions of the IVPs*

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ \varepsilon y_2^{*'} + a(x)y_2^* - b(x)y_1^* = f(x) + c(x)u_{0_1}, \quad x \in \Omega, \\ y_1^*(0) = \alpha, \quad y_2^*(0) = \beta. \end{cases} \tag{15}$$

and

$$\begin{cases} y_1^{1*'} - y_2^{1*} = 0, \\ \varepsilon y_2^{1*'} + a(x)y_2^{1*} - b(x)y_1^{1*} = f(x) + c(x)u_{0_1}, \quad x \in \Omega, \\ y_1^{1*}(0) = \alpha + O(\varepsilon), \quad y_2^{1*}(0) = \beta. \end{cases} \tag{16}$$

then, $||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon.$

Theorem 3.10

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ *be the solution of the IVPs (15). Further, let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVPs (16) after applying the Euler finite difference scheme as given in (13). Then,*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq C\varepsilon + CN^{-1}\ln N, \quad \text{for} \quad 0 \leq i \leq N/2 \quad \text{and} \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

The BVPs (7)-(8) is equivalent to the following IVPs

$$\begin{cases} \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f^*(x), x \in \Omega, \\ y_2(0) = q^*,\ y_2'(0) = -q. \end{cases} \tag{17}$$

where q^* is the exact value of the solution of the BVP (7)-(8) at $x = 0$. Because of uniqueness of the solutions of the IVP (17) and the BVPs (7)-(8), we have the following result on the error estimate for the inner region problem.

Theorem 3.11

Let $y_2^\star(x_i)$ *be the solution of the BVPs (7)-(8). Further, let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVPs (13).Then,*

$$|y_2^\star(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad \text{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Theorem 3.12

Let $\bar{y}(x)$ *be the solution of the BVPs (3) and let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVPs (13). Then,*

$$|y_2(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad \text{for} \ 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Outer region

Theorem 3.13

Let $y_2(x_i)$ be the solution of the BVPs (10) and $y_{2,i}$ be its numerical solution given by (14).Then,

$$|y_2(x_i) - y_{2,i}| \leq CN^{-1} \ln N, \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_{\tau}^{N/2}.$$

Theorem 3.14

Let $y_2^{\star}(x_i)$ be the solution of the BVPs (7)-(8) and $y_{2,i}$ be the numerical solution of the BVPs (10) after applying the FD scheme as given in (14).Then,

$$|y_2^{\star}(x_i) - y_{2,i}| \leq C\varepsilon + CN^{-1} \ln N, \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_{\tau}^{N/2}.$$

Theorem 3.15

Let $\bar{y}(x)$ be the solution of the BVPs (3) and $y_{2,i}$ be the numerical approximation obtained for $y_2(x_i)$ for the BVP (10) after applying the FD scheme as given in (14). Then,

$$|y_2(x_i) - y_{2,i}| \le C\varepsilon + CN^{-1}\ln N, \quad \text{for} \quad 0 \le i \le N/2,\ x_i \in \bar{\Omega}_\tau^{N/2}.$$

Problem Description

This numerical method is also applied for the following class of problems.

Problem class (II). (Third Order Reaction-Diffusion Equations)

$$\begin{cases} \text{Find} \quad y \in C^3(\Omega) \cap C^2(\bar{\Omega}) \text{ such that} \\ -\varepsilon y'''(x) + b(x)y'(x) + c(x)y(x) = f(x),\ x \in \Omega, \\ y(0) = p, \quad y''(0) = q, \quad y''(1) = r. \\ b(x) \geq \beta, \quad \beta > 0, \quad 0 \geq c(x) \geq -\gamma, \quad \text{for some } \gamma > 0 \\ \beta - 2\gamma \geq \gamma' \quad \text{for some } \gamma' > 0. \end{cases}$$

where $0 < \varepsilon \ll 1$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions.

Problem class (III).Fourth Order Convection-Diffusion Equations

$$\begin{cases} \text{Find} \quad y \in C^4(\Omega) \cap C^3(\bar{\Omega}) \text{ such that} \\ \varepsilon y^{iv}(x) + a(x)y'''(x) - b(x)y''(x) + c(x)y(x) = f(x), \ x \in \Omega, \\ y(0) = p, \quad y'''(0) = q, \quad y(1) = r, \quad -y''(1) = s, \quad a(x) \geq \alpha, \quad \alpha > 0, \\ 0 \geq c(x) \geq -\gamma, \quad \alpha > 3\gamma, \text{ for some } \gamma > 0, \end{cases}$$

where $0 < \varepsilon \ll 1, \ \Omega = (0,1), \ \bar{\Omega} = [0,1] \ a(x), \ b(x), \ c(x)$ are sufficiently smooth functions.

Problem class (IV). Fourth Order Reaction-Diffusion Equations

$$\begin{cases} \text{Find} \quad y \in C^4(\Omega) \cap C^3(\bar{\Omega}) \text{ such that} \\ -\varepsilon^2 y^{iv}(x) + b(x)y''(x) + c(x)y(x) = f(x), \ x \in \Omega, \\ y(0) = p, \quad -y'''(0) = q, \quad y'(1) = r, \quad -y'''(1) = s. \\ b(x) \geq \beta, \quad \beta > 0, \quad 0 \geq c(x) \geq -\gamma, \beta \geq \gamma, \quad \text{for some } \gamma > 0. \end{cases}$$

where $0 < \varepsilon \ll 1$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions.
Using Newton's method of quasilinearization, this method is also applied to nonlinear problems of above mentioned type.

Conclusions

- In [20], both inner and outer region problems are BVP, whereas in our case the inner region problem is an IVP and the outer region problem is a BVP. Naturally IVP can be treated more easily compared with BVP.
- Though the present method yields almost the same order of convergence as given in [20], it produces very good reduction on the maximum-pointwise error compared with .
- The main advantage of this method is that due to decoupling, the size of the matrix to be inverted is reduced from $2N - 1$ to $N - 1$. This results in a good reduction of the computation time.

Schwarz method

Existing Work

- The authors of [28, 33] constructed an overlapping Schwarz method for Singularly Perturbed second order Convection-Diffustion equations.
- The authors used Classical Finite Difference scheme to find the numerical solution for the problem given in [28, 33].
- It is shown that the numerical approximations generated by this method fail to converge parameter-uniformly to the solution of the continuous problem.

Statement of the Problem

Consider the following Singularly Perturbed Convection-Diffusion problem as given in [28, 33].

$$Ly := \quad -\varepsilon y''(x) + a(x)y'(x) = f(x),\; x \in \Omega = (0,1), \tag{18}$$

$$y(0) = q_0, \quad y(1) = q_1, \tag{19}$$

where q_0, q_1 are given constants and the functions y, a and f $\in C^{(3)}(\bar{\Omega})$ with $a(x) > \alpha,\ \alpha > 0, and\, 0 < \varepsilon \ll 1$.

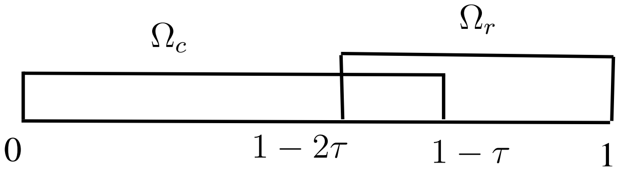

Figure: The overlapping subdomains of Ω .

First, we split the domain into two overlapping subdomains as

$$\Omega_c = (0, 1-\tau) \quad \text{and} \quad \Omega_r = (1-2\tau, 1),$$

where the subdomain parameter τ is an appropriate constant given by $\tau = \min\left\{\frac{1}{3}, \frac{2\varepsilon}{\alpha}\ln N\right\}$.

The continuous Schwarz method generates a sequence of iterates $\{y^{[k]}\}$, which converges as $k \to \infty$ to the exact solution y. For $k \geq 1$, the iterates $y^{[k]}(x)$ are defined by

$$y^{[k]}(x) = \begin{cases} y_c^{[k]}(x) & \text{for } \; x \in \bar{\Omega}_c, \\ y_r^{[k]}(x) & \text{for } \; x \in \bar{\Omega}_r \backslash \bar{\Omega}_c, \end{cases}$$

The iterative process is defined as follows:

$$y^{[0]}(x) \equiv 0, \;\; 0 < x < 1, \;\; y^{[0]}(0) = y(0), \;\; y^{[0]}(1) = y(1).$$

where, for $k = 1$, $y_r^{[1]}(x)$ and $y_c^{[1]}(x)$ are the solutions of the problems

$$\begin{aligned} Ly_r^{[1]}(x) &= f \text{ in } \Omega_r, \ y_r^{[1]}(1-2\tau) = y^{[0]}(1-2\tau) = 0, \ y_r^{[1]}(1) = y(1) \\ \text{and } Ly_c^{[1]}(x) &= f \text{ in } \Omega_c, \ y_c^{[1]}(0) = y(0), \ y_c^{[1]}(1-\tau) = y_r^{[1]}(1-\tau). \end{aligned}$$

and, for all $k > 1$, $y_p^{[k]}, p = \{c, r\}$ are the solutions of the problems

$$\begin{aligned} Ly_r^{[k]}(x) &= f \text{ in } \Omega_r, y_r^{[k]}(1-2\tau) = y^{[k-1]}(1-2\tau), \ y_r^{[k]}(1) = y(1) \\ \text{and } \ Ly_c^{[k]}(x) &= f \text{ in } \Omega_c, \ y_c^{[k]}(0) = y(0), \ y_c^{[k]}(1-\tau) = y_r^{[k]}(1-\tau). \end{aligned}$$

Let $\Omega_p = (d, e), \quad \bar{\Omega}_p = [d, e], \quad p = \{c, r\}$. The BVP (18)-(19) satisfies the following maximum principle on each $\bar{\Omega}_p$.

Theorem 5.1

(Maximum Principle). *Consider the BVPs (18)-(19). Let* $y(d) \geq 0, y(e) \geq 0,\ Ly(x) \geq 0$, *for* $x \in \Omega_p$. *Then,* $y(x) \geq 0, \forall x \in \bar{\Omega}_p$.

An immediate consequence of this is the following stability result.

Lemma 5.2

(Stability Result). *If* $y(x)$ *is the solution of the BVPs (18)-(19) then*

$$||y(x)|| \leq [\max\{|y(d)|, |y(e)|\} + \frac{1}{\alpha}||f||], \forall x \in \bar{\Omega}_p.$$

Discrete Schwarz method

- The continuous overlapping Schwarz method is descretized by introducing uniform meshes on each subdomains.
- In each subdomain, $\Omega_p = (d, e), \quad p = \{c, r\}$, construct a uniform mesh $\bar{\Omega}_p^N = \{d = x_0 < x_1 < x_2 < \cdots < x_N = e\}$ with $h_p = x_i - x_{i-1} = (e - d)/N$.
- In this proposed scheme we use the central finite difference scheme on a uniform mesh in the subdomain Ω_r and the mid point difference scheme on a uniform mesh in the subdomain Ω_c.

Then in each subdomain Ω_p^N, $\quad p = \{c, r\}$, the corresponding discretization is,

$$L^N Y_c(x_i) = -\varepsilon\delta^2 Y_c(x_i) + a_{i-1/2} D^- Y_c(x_i) = f_{i-1/2}, \quad i = 1, \dots, N-1, \tag{20}$$

$$L^N Y_r(x_i) = -\varepsilon\delta^2 Y_r(x_i) + a_i D^0 Y_r(x_i) = f_i, \quad i = 1, \dots, N-1. \tag{21}$$

where $\delta^2 Y_p(x_i) = \dfrac{1}{h_p^2}\left(Y_p(x_{i+1}) - 2Y_p(x_i) + Y_p(x_{i-1})\right)$, $D^0 Y_r(x_i) = \dfrac{Y_r(x_{i+1}) - Y_r(x_{i-1})}{2h_r}$, $D^- Y_c(x_i) = \dfrac{Y_c(x_i) - Y_c(x_{i-1})}{h_c}$, $a_{i-1/2} \equiv a((x_{i-1} + x_i)/2)$, and $a_i \equiv a(x_i)$; Similarly for $f_{i-\frac{1}{2}}$, f_i.

The sequence of discrete Schwarz iterates $Y^{[k]}$ are defined as follows:

$$Y^{[0]}(x_i) \equiv 0, \quad 0 < x_i < 1, \quad Y^{[0]}(0) = y(0), \quad Y^{[0]}(1) = y(1).$$

For all $k \geq 1$, the iterates $Y^{[k]}$ are defined by

$$Y^{[k]}(x_i) = \begin{cases} Y_c^{[k]}(x_i), x_i \in \bar{\Omega}_c^N, \\ Y_r^{[k]}(x_i), x_i \in \bar{\Omega}_r^N \backslash \bar{\Omega}_c. \end{cases} \tag{22}$$

where $Y_p^{[k]}, p = \{r, c\}$ are defined to be the solutions of the problems

$$\begin{aligned} L^N Y_r^{[k]}(x_i) &= f_i, x_i \in \Omega_r^N,\ Y_r^{[k]}(1-2\tau) = \bar{Y}^{[k-1]}(1-2\tau),\ Y_r^{[k]}(1) = y(1), \\ L^N Y_c^{[k]}(x_i) &= f_{i-\frac{1}{2}}, x_i \in \Omega_c^N,\ Y_c^{[k]}(0) = y(0),\ Y_c^{[k]}(1-\tau) = \bar{Y}_r^{[k]}(1-\tau). \end{aligned}$$

where $\bar{Y}^{[k]}$ denotes the piecewise linear interpolant of $Y^{[k]}$ on the mesh $\bar{\Omega}^N = (\bar{\Omega}_r^N \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c^N$.

- This iterative process is repeated until successive iterates are sufficiently close at each point of $\bar{\Omega}^N$, in the sense that they satisfy the stopping criterion $||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \leq 10^{-14}$.

Theorem 5.3

(Discrete maximum principle) Assume that $Y(x_0) \geq 0$ and $Y(x_N) \geq 0$ then $L^N Y(x_i) \geq 0, \forall x_i \in \Omega_p^N$ implies that $Y(x_i) \geq 0, \forall x_i \in \bar{\Omega}_p^N$.

An immediate consequence of this lemma is the following stability result.

Lemma 5.4

If $Y(x_i)$ is any mesh function then for all $x_i \in \bar{\Omega}_p^N$,

$$|Y(x_i)| \leq C \max\{|Y(x_0)|, |Y(x_N)|, ||L^N Y||_{\Omega_p^N}\}.$$

In Lemma 5.5 we show that the subdomain iterations converge to the solution of continuous problem.

Lemma 5.5

Let y be the solution of (18)-(19) and let $Y^{[k]}$ be the k^{th} iterate of the discrete Schwarz method. Then, there are constants C such that

$$||Y^{[k]} - y||_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-1}\ln^3 N,$$

where C is independent of k and N.

In the following lemma we show that the iterative process converges much faster than is shown in Lemma 4.1.

Lemma 5.6

Let $Y^{[k]}$ be the k^{th} iterate of the discrete Schwarz method. Then there exists some C such that

$$||Y^{[k+1]} - Y^{[k]}||_{\bar{\Omega}^N} \leq C\nu^k \quad \text{where} \quad \nu = \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} < 1,$$

and C is independent of k and N. Furthermore if $\tau = \dfrac{2\varepsilon}{\alpha} \ln N$ *then* $\nu \leq 2N^{-1/2}$.

The following theorem which is the main result of this paper, combining Lemmas 5.5 and 5.6 to prove that, two iterations are sufficient to attain first order convergence.

Theorem 5.7

Let y be the solution of (18)-(19) and $Y^{[k]}$ be the k^{th} iterate of the discrete Schwarz method. If $\tau = \frac{2\varepsilon}{\alpha} \ln N$ and $N > 2$, then

$$||Y^{[k]} - y||_{\bar{\Omega}^N} \le CN^{-k/2} + CN^{-1} \ln^3 N,$$

where C is independent of k and N.

Numerical Experiments

We present two examples to illustrate the theoretical results for the BVPs (18)-(19). Let Y_j^N be a Schwarz numerical approximation for the exact solution y_j on the mesh Ω^N and N is the number of mesh points.Let Y^N be a numerical approximation for the exact solution y on the mesh Ω^N and N is the number of mesh points. For a finite set of values of $\varepsilon = \{2^0 \cdots 2^{-31}\}$, we compute the maximum point-wise errors

$$D_\varepsilon^N = ||Y^N - y||_{\Omega^N}, \quad D^N = \max_\varepsilon D_\varepsilon^N.$$

From these quantities the order of convergence are computed from

$$p^N = log_2 \left\{ \frac{D^N}{D^{2N}} \right\}.$$

Example 5.8

$$-\varepsilon y''(x) + y'(x) = 0, \quad x \in \Omega$$
$$y(0) = 0, \quad y(1) = 1.$$

In [28, 33], the authors consider the example 5.8 as a test problem. The exact maximum pointwise errors at each of the mesh points in $\bar{\Omega}^N$ are given in Table 1, for various values of ε and N (as in [33]).

Table: *Computed nodal maximum pointwise errors D_ε^N, when the standard classical finite difference scheme used in Schwarz method with a uniform mesh in each subdomain is applied to example 5.8 as in [33].*

Number of mesh points N							
ε	8	16	32	64	128	256	512
2^0	4.78e-03	2.45e-03	1.24e-03	6.25e-04	3.14e-04	1.57e-04	7.86e-05
2^{-4}	9.97e-02	5.89e-02	3.96e-02	3.69e-02	2.27e-02	1.35e-02	7.53e-03
2^{-8}	6.84e-01	2.99e-01	7.86e-02	2.54e-02	1.40e-02	7.88e-03	4.44e-03
2^{-12}	9.59e-01	8.82e-01	6.00e-01	2.23e-01	5.38e-02	1.40e-02	5.03e-03
2^{-16}	9.83e-01	9.88e-01	9.60e-01	8.28e-01	4.91e-01	1.65e-01	3.98e-02
2^{-20}	9.84e-01	9.96e-01	9.96e-01	9.87e-01	9.40e-01	7.63e-01	4.08e-01
2^{-24}	9.84e-01	9.96e-01	9.99e-01	9.99e-01	9.96e-01	9.81e-01	9.17e-01
2^{-28}	9.84e-01	9.96e-01	9.99e-01	1.00e+00	1.00e+00	9.99e-01	9.94e-01
2^{-34}	**9.84e-01**	**9.96e-01**	**9.99e-01**	**1.00e+00**	**1.00e+00**	**1.00e+00**	**9.99e-01**

Table: *Computed Iteration counts for various of ε and N, hen the standard classical finite difference scheme used in Schwarz method with a uniform mesh in each subdomain is applied to example 5.8 as in [33].*

	Number of mesh points N							
ε	8	16	32	64	128	256	512	1024
2^{0}	16	16	16	16	16	16	16	16
2^{-4}	16	10	7	6	5	4	4	4
2^{-8}	82	70	34	12	8	6	4	4
2^{-12}	110	186	254	197	97	40	14	7
2^{-16}	112	207	399	705	856	585	283	123
2^{-20}	112	209	414	838	1629	2669	2828	1774
2^{-24}	112	209	415	848	1727	3427	6351	9574
2^{-28}	112	209	415	849	1733	3489	6887	13194

Example 5.9

$$-\varepsilon y''(x) + (1 + x)y'(x) = e^{-1/\varepsilon}, \quad x \in \Omega$$
$$y(0) = 0, \quad y(1) = 1$$

- The authors of [28] and [33] conclude that the discrete Schwarz iterates constructed by them did not converge to the solutions of continuous problem and the method does not produce ε -uniform convergence.

- But it is not true in my research case, because of the proposed scheme, the discrete Schwarz iterates converges ε-uniformly to the solutions of the continuous problem. We checked this by examples 5.8 and 5.9.
- Numerical approximations and iteration counts (k) are given in Tables. Also we have presented the graph of the exact and numerical solutions of examples 5.8 for various values of ε and N.

Table: *Computed nodal maximum pointwise errors D_ε^N, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.8.*

Number of mesh points N				
ε	64	128	256	512
2^0	**2.3874e-001**	**2.3448e-001**	**2.3236e-001**	**2.3130e-001**
2^{-4}	2.7424e-005	2.5222e-005	2.4188e-005	2.3687e-005
2^{-8}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-12}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-16}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-20}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-24}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-28}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-32}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
p^N	2.5975e-002	1.3103e-002	6.5965e-003	3.3096e-003
k	2	2	2	2

Table: *Computed nodal maximum pointwise errors D_ε^N, when the new scheme proposed in Schwarz method with a uniform mesh in each subdomain is applied to example 5.9.*

Number of mesh points N				
ε	64	128	256	512
2^{0}	**2.4502e-001**	**2.4126e-001**	**2.3938e-001**	**2.3845e-001**
2^{-4}	2.7453e-005	2.5251e-005	2.4217e-005	2.3715e-005
2^{-8}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-12}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-16}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-20}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-24}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-28}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
2^{-32}	7.7298e-008	4.3352e-009	2.5390e-010	1.5279e-011
p^N	2.2311e-002	1.1286e-002	5.6158e-003	2.8464e-003
k	2	2	2	2

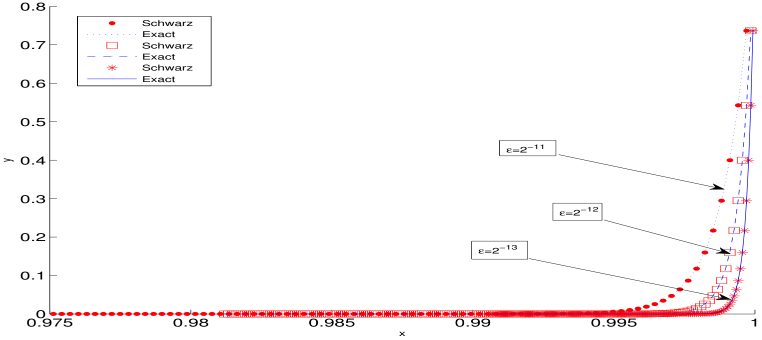

Figure: Comparison of the exact and the numerical solution of example 5.8 with $N = 64$ (within the layer region).

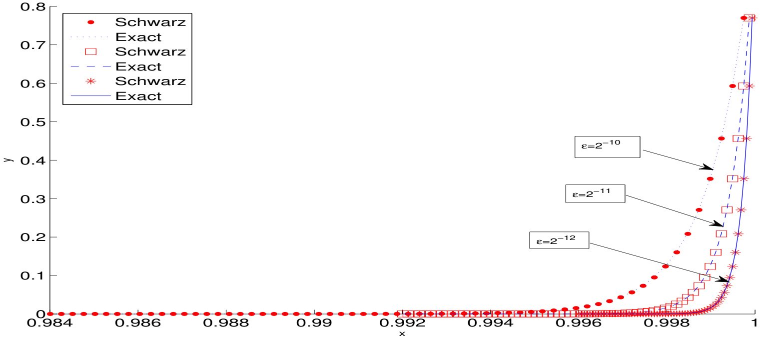

Figure: Comparison of the exact and the numerical solution of example 5.9 with $N = 64$ (within the layer region).

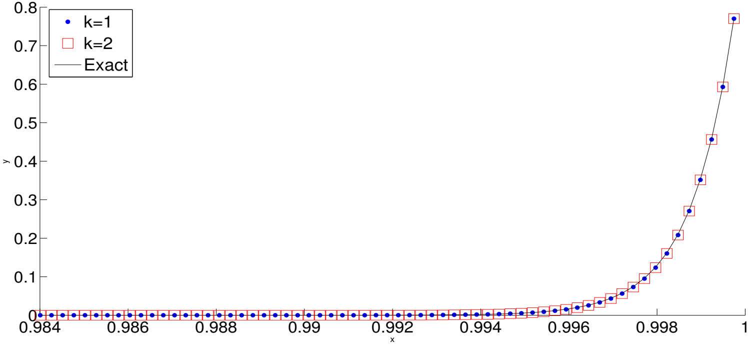

Figure: Comparison of the exact and the numerical solution at iterations, $k = 1, 2$ of example 5.8 with $N = 64$, $\varepsilon = 2^{-10}$, $\alpha = 1$ (within the layer region).

This Schwarz method is also applied for the following types of problems.
Problem class (V): (Third order convection-diffusion equation)

$$\begin{cases} \text{Find} \quad y \in C^3(\Omega) \cap C^2(\bar{\Omega}) \\ -\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \ x \in \Omega, \\ y(0) = q_1, \quad y'(0) = q_2, \quad y'(1) = q_3. \\ a(x) \geq \alpha', \ \alpha' > 0, \quad b(x) \geq 0, \quad 0 \geq c(x) \geq -\gamma \quad \gamma > 0, \\ \alpha' > 8\gamma, \ \text{ for some } \gamma > 0, \end{cases}$$

where $0 < \varepsilon \ll 1$, $\Omega = (0,1)$, $\bar{\Omega} = [0,1]$, $a(x)$, $b(x)$, $c(x)$, $f(x)$ are sufficiently smooth functions.

Problem class (VI): (Fourth order convection-diffusion equation)

$$\begin{cases} \text{Find} \quad y \in C^4(\Omega) \cap C^2(\bar{\Omega}) \\ -\varepsilon y^{iv}(x) + a(x)y'''(x) + b(x)y''(x) + c(x)y(x) = -f(x),\ x \in \Omega, \\ y(0) = q_1, \quad y''(0) = -q_2, \quad y(1) = q_3 \quad y''(1) = -q_4. \\ a(x) \geq \alpha,\ \alpha > 0,\ b(x) \geq 0,\ 0 \geq c(x) \geq -\gamma,\ \alpha > 8\gamma,\ \text{for some } \gamma > 0, \end{cases}$$

where $0 < \varepsilon \ll 1$, $\Omega = (0, 1)$, $\bar{\Omega} = [0, 1]$, $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions.

Problem class(VIII). (Second order system of convection-diffusion equations)

$$\begin{cases} \text{Find} \quad y_1, y_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega) \quad \text{such that} \\ -\varepsilon y_1'' + a_1(x)y_1' + b_{11}(x)y_1 + b_{12}(x)y_2 = f_1(x), \\ -\varepsilon y_2'' + a_2(x)y_2' + b_{21}(x)y_1 + b_{22}(x)y_2 = f_2(x), \; x \in \Omega \\ y_1(0) = q_1, \quad y_1(1) = q_2, \quad y_2(0) = q_3, \quad y_2(1) = q_4. \end{cases}$$

with the following conditions,

$$a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \quad \alpha_1 \neq \alpha_2, \quad b_{12}(x) \leq 0,$$
$$b_{21}(x) \leq 0 \text{ and } \sum_{j=1}^{2} b_{ij}(x) \geq \beta > 0, \; \forall x \in \bar{\Omega},$$

where $\Omega = (0,1), \bar{\Omega} = [0,1], \; 0 < \varepsilon \ll 1,$ the functions $a_i(x), \; b_{ij}(x)$ and $f_i(x), \; i,j = 1,2$ are sufficiently smooth on $\bar{\Omega}$.

Conclusions

- With or without domain decomposition generally classical finite difference scheme (CFD) applied on scalar SPPs works well. But in the case of CFD scheme used in Schwarz method as in [33], it's solution doesn't converge to the exact solution. This is illustrated in Table 1. This shows the poor efficiency of schwarz method used with CFD scheme.
- Generally central finite difference and mid point difference schemes yield second order convergence for scalar SPPs. When they are used in Schwarz method which discussed earlier, yields only first order convergence. The proposed schemes used in the Schwarz method, helps us to overcome fundamental difficulty mentioned in [28, 33]. In [28, 33], the authors used the same scheme in both the domains Ω_r and Ω_c whereas in my reseach case, the different schemes are used in each subdomain Ω_r and Ω_c.

- From theorem 5.7 it can be easily identified in which iterations, the Schwarz iterate terminates.
- From the above examples number of iterations taken by using proposed scheme in this method is not more than two which is very much reduced when comparing iteration counts presented in [33].
- Since this method yields parameter-uniform second order convergence for Weakly coupled system, Third and fourth order Convection-Diffusion equations, is more suitable for the above mentioned problems.

List of articles Publication

- J. Christy Roja and A. Tamilselvan, ***Shooting method for singularly perturbed fourth order ordinary differential equations of reaction-diffusion type, International Journal of Computational Methods, Vol. 10, No. 6 (2013).***
- J. Christy Roja and A. Tamilselvan, ***Numerical method for singularly perturbed third order ordinary differential equations of convection-diffusion type, Numerical Mathematics Theory Methods Application Vol. 7, No. 3, pp. 265-287.***
- J. Christy Roja and A. Tamilselvan, *A parameter uniform second order Schwarz method for singularly perturbed second order ordinary differential equations of convection diffusion type.* (Revised and resubmitted to the journal of Numerical algorithm).

List of articles communicated for publication

- J. Christy Roja and A. Tamilselvan, *Numerical method for singularly perturbed third order ordinary differential equations of reaction-diffusion type.*
- J. Christy Roja and A. Tamilselvan, *Numerical method for singularly perturbed fourth order ordinary differential equations of convection-diffusion type.*
- J. Christy Roja and A. Tamilselvan, *A parameter uniform second order Schwarz method for singularly perturbed third order ordinary differential equations of convection diffusion type.*
- J. Christy Roja and A. Tamilselvan, *A parameter uniform second order Schwarz method for singularly perturbed fourth order ordinary differential equations of convection diffusion type.*
- J. Christy Roja and A. Tamilselvan, *A parameter-uniform second order Schwarz method for a weakly coupled system of singularly perturbed convection diffusion equations.*

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Thank You