Plan of talk

1. Introduction
2. Mathematical model for a turning point problem
3. Applications
4. Problems studied in the thesis
5. References
The birth of the SPPs was introduced by Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904 and it was reported in the proceedings of the conference.

Many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems are described by differential equations involving large or small parameters.

The solutions of SPPs have non-uniform behavior. That is, there are thin layer(s) (boundary layer region) where the solution varies rapidly while away from the layer(s) (outer region) the solution behaves regularly and varies slowly.
Let $P_\varepsilon$ denote the original problem and $u_\varepsilon$ be its solution. Let $P_0$ denote the reduced problem of $P_\varepsilon$ (setting $\varepsilon = 0$ in $P_\varepsilon$) and $u_0$ be its solution. Then the problem $P_\varepsilon$ is called a **Singular Perturbation Problem (SPP)** if and only if $u_\varepsilon$ does not converge uniformly to $u_0$ in the entire domain of the definition of the problem. Otherwise the problem is called **Regular Perturbation Problem (RPP)**.

**Example 1 (Regular Perturbation Problem)**

\[
P_\varepsilon: \begin{cases} u_\varepsilon'(x) = -\varepsilon u_\varepsilon(x), & x \in (0, 1], \\ u_\varepsilon(0) = 1, & 0 < \varepsilon \ll 1. \end{cases}
\]

\[
P_0: u_0'(x) = 0, \ x \in (0, 1], \ u_0(0) = 1,
\]
Example 2 (Singular Perturbation Problem)

\[
P_\varepsilon : \begin{cases} \\
\varepsilon u_\varepsilon'(x) = -u_\varepsilon(x), & x \in (0, 1], \\
u_\varepsilon(0) = 1, & 0 < \varepsilon \ll 1. 
\end{cases}
\]

\[
P_0 : \quad u_0(x) = 0, \quad x \in [0, 1],
\]

The exact solution is given by

\[ u_\varepsilon(x) = \exp(-x/\varepsilon). \]

Note that,

\[
\lim_{\varepsilon \to 0} \lim_{x \to 0} u_\varepsilon(x) = 1, \\
\lim_{x \to 0} \lim_{\varepsilon \to 0} u_\varepsilon(x) = 0.
\]

That is, \( u_\varepsilon(x) \) does not converge uniformly to the reduced problem solution on \([0, 1]\).
The solution changes very rapidly near the neighborhood of $x = 0$. This neighborhood is called a boundary layer.
Often these mathematical problems are extremely difficult (or even impossible) to solve exactly and in these circumstances approximate solutions are necessary. One can obtain an approximate solution through the use of perturbation methods.

In general, regular numerical methods like Euler method, Runge Kutta methods, finite difference methods, etc cannot be applied to these SPPs.

For Example
The disadvantage in the classical numerical methods (finite difference/finite element) is due to the nature of the coefficients. That is, inaccurate solution due to perturbation parameter.

Classical numerical methods on equidistant grids yield satisfactory numerical solution for singularly perturbed boundary value problems only if one uses an unacceptably large number of grid points.

In order to overcome this difficulty we apply numerical methods on appropriate meshes like Shishkin mesh, Bakhvalov mesh, Bakhvalov Shishkin mesh etc.
The main difference between singular perturbation problem and singularly perturbed turning point problem is the coefficient of the convection term vanishes inside the domain of the differential equation.

If the turning point occur at the interior of the domain, then the problem is called as an interior turning point problem, otherwise it is a boundary turning point problem.

If the velocity distribution is linear, then the problem is known as a simple turning point problem, otherwise it is a multiple turning point problem.
Mathematical model for a turning point problem

Consider the one dimensional equation [3] which describes a quantum mechanical particle in a potential $V(x)$

$$\left(-\varepsilon^2 \frac{d^2}{dx^2} + V(x) - E\right) y(x) = 0,$$

where $V(x)$ is the potential energy of the particle and $E$ is the total energy of the particle.

- For this equation, $Q(x) = V(x) - E$, so $Q(x)$ vanishes at points where $V(x) = E$ and these are called turning points.
- The classical orbit of a particle in the potential $V(x)$ is confined to the regions where $V(x) \leq E$.
- The particle moves until it reaches a point where $V = E$ and then it stops, turns around and moves off in the opposite direction.
Applications

SPTPPs occur in the modelling of following problems.

- Modeling of steady and unsteady viscous flow problems with large Reynolds number
- Navier Stokes flows with large Reynolds numbers
- Magneto-hydrodynamic duct problems at high Hartman numbers
- Heat transport problem with large Peclet numbers
- One dimensional version of stationary convection-diffusion problems with a dominant convective term
- Speed field that changes its sign in the catch basin
- Geophysics and modeling thermal boundary layers in laminar flow.
A typical linear turning point problem in one dimension[12] is given by

**Example 3**

\[-\varepsilon u''(x) + xb(x)u'(x) + c(x)u(x) = f(x), \ x \in (-1, 1), \ u(-1) = u(1) = 0\]

under the following assumptions:

(i) \( b(x) \neq 0 \) on \([-1, 1]\)  
(ii) \( c(x) \geq 0, \ c(0) > 0 \).

- The location of any boundary layer(s) depends on the sign of the convection term.
- From our experience, we expect a boundary layer at \( x = -1 \) if the coefficient of the convection term \( xb(x) \) is negative at \( x = -1 \), and a boundary layer at \( x = 1 \) if the same coefficient is positive at \( x = 1 \).
- If \( b(x) \) is positive on \([-1, 1] \), we have \( \left. xb(x) \right|_{x=-1} < 0 \) and \( \left. xb(x) \right|_{x=1} > 0 \).  
- Consequently, if \( b \) is positive on \([-1, 1] \), then the solution \( u \) has two boundary layers at \( x = 1 \) and \( x = -1 \) otherwise the solution has interior layer at \( x = 0 \).
Example 4 (Exhibiting layers at the boundary)

Consider the BVP

\[ \varepsilon u''(x) - 2(2x - 1)u'(x) - 4u(x) = 0 \quad \forall \ x \in (0, 1) \]
\[ u(0) = 1, \quad u(1) = 1 \]

The exact solution is given by

\[ u(x) = e^{-2x(1-x)/\varepsilon} \]
Figure: Exact solution of example 4 for $\varepsilon = 2^{-2}$ to $\varepsilon = 2^{-10}$ and $N = 1024$
Example 5 (Exhibiting layers at the interior)

Consider the BVP

\[ \varepsilon u''(x) + 2xu'(x) = 0 \quad \forall \quad x \in (-1, 1) \]
\[ u(-1) = -1, \quad u(1) = 1 \]

The exact solution is given by

\[ u(x) = \text{erf}(x/\sqrt{\varepsilon}) \]
**Figure:** Exact solution of example 5 for $\varepsilon = 2^{-2}$ to $\varepsilon = 2^{-10}$ and $N = 1024$
In the present thesis, motivated by the works of [1, 4, 6, 9, 11, 16, 17, 19, 20, 21], two methods are given namely

- Parameter Uniform Finite Difference Method (PUFDM)
- Variable Mesh Spline Approximation Method (VMSAM)

for various singularly perturbed turning point problems.

The PUFDM and VMSAM are discussed for Problem class I, whereas the PUFDM is applied for Problem classes II to V.
Problem class I: Second order SPTPPs with Robin boundary conditions

Find \( u \in C^1(\tilde{\Omega} = [-1, 1]) \cap C^2(\Omega = (-1, 1)) \) such that

\[
Lu \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad \forall \ x \in \Omega \tag{1}
\]

with Robin boundary conditions

\[
B_1 u(-1) = \beta_1 u(-1) - \varepsilon \beta_2 u'(-1) = A \tag{2}
\]

\[
B_2 u(1) = \gamma_1 u(1) + \varepsilon \gamma_2 u'(1) = B
\]

and the assumptions

\[
\begin{cases}
  a(0) = 0, \quad a'(0) < 0, \quad |a(x)| \leq \alpha_0 > 0, \quad 0 < \beta_0 \leq b(x), \\
  \alpha_0 < \beta_0, \quad |a'(x)| \geq \frac{|a'(0)|}{2} \forall x \in \tilde{\Omega}, \ eta_1, \beta_2 \geq 0, \\
  \beta_1 - \varepsilon \beta_2 > 0, \quad \gamma_2 \geq 0 \& \gamma_1 > 0. \tag{3}
\end{cases}
\]

where \( \varepsilon (0 < \varepsilon \ll 1) \) is a small positive parameter, \( a(x), b(x) \) and \( f(x) \) are sufficiently smooth functions on \( \tilde{\Omega} \).
Theorem 6 (Minimum Principle)

Let $L$ be the differential operator defined in (1) and $v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. If $B_1 v(-1) \geq 0$, $B_2 v(1) \geq 0$ and $Lv \leq 0 \ \forall \ x \in \Omega$, then $v(x) \geq 0 \ \forall \ x \in \overline{\Omega}$.

Lemma 7 (Stability Result)

If $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, then

$$|u(x)| \leq C \max \{ \max \{|B_1 u|, |B_2 u|\}, \|Lu\|_{x \in \Omega} \}, \forall x \in \overline{\Omega}.$$
From now on we shall denote the subdomains of $\bar{\Omega} = [-1, 1]$ as $\Omega_1 = [-1, -\delta]$, $\Omega_2 = [-\delta, \delta]$ and $\Omega_3 = [\delta, 1]$, $0 < \delta \leq 1/2$. The choice of $\delta = 1/2$ can be found in [4].

**Lemma 8**

Let $u$ be the solution of (1)-(3). Then

$$
\|u^{(k)}\| \leq C \varepsilon^{-(k)} \max\{\|f\|, \|u\|\}, \quad k = 1, 2
$$

$$
\|u^{(3)}\| \leq C \varepsilon^{-(3)} \max\{\|f\|, \|f'\|, \|u\|\},
$$

$\forall x \in \Omega_1 \cup \Omega_3$, where $C$ depends on $\|a\|, \|a'\|, \|b\|$ and $\|b'\|$. 
The following lemma gives estimates for $u$ and its derivatives in the interval $\Omega_2$ which includes the turning point $x = 0$.

**Lemma 9**

*Let $u$ be the solution of (1)-(3). Then

$$||u^{(k)}(x)|| \leq C, \quad \forall x \in \Omega_2,$$

where $C$ depends on $||a||, ||a'||, ||b||, ||b'||, ||f||, ||f'||$ and $\beta$.***
To derive $\varepsilon$ - uniform error estimates we require sharper bounds of the solution and its derivatives. For this we use Shishkin decomposition of the solution $u$ as

$$ u = v + w. $$

Here $v$ is the solution of the problem

$$ L v = f $$

$$ \beta_1 v(-1) - \varepsilon \beta_2 v'(-1) = \beta_1 v_0(-1) - \varepsilon \beta_2 v'_0(-1) + \varepsilon (\beta_1 v_1(-1) - \varepsilon \beta_2 v'_1(-1)), $$

$$ \gamma_1 v(1) + \varepsilon \gamma_2 v'(1) = \gamma_1 v_0(1) + \varepsilon \gamma_2 v'_0(1) + \varepsilon (\gamma_1 v_1(1) + \varepsilon \gamma_2 v'_1(1)) $$

where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$.

Also $v_0$ and $v_1$ are defined respectively, to be the solutions of the reduced problem:

$$ a v'_0 - b v_0 = f \quad \text{and} \quad a v'_1 - b v_1 = -v''_0 $$

and $v_2$ is the solution of the problem similar to that defining $u$

$$ L v_2 = -v''_1, $$

$$ \beta_1 v_2(-1) - \varepsilon \beta_2 v'_2(-1) = 0, \quad \gamma_1 v_2(1) + \varepsilon \gamma_2 v'_2(1) = 0. $$
The singular component \( w \) is the solution of the homogeneous problem

\[
Lw = 0,
\]

(7)

\[
\beta_1 w(-1) - \varepsilon \beta_2 w'(-1) = (\beta_1 u(-1) - \varepsilon \beta_2 u'(-1)) - (\beta_1 v(-1) - \varepsilon \beta_2 v'(-1)),
\]

\[
\gamma_1 w(1) + \varepsilon \gamma_2 w'(1) = (\gamma_1 u(1) + \varepsilon \gamma_2 u'(1)) - (\gamma_1 v(1) + \varepsilon \gamma_2 v'(1)).
\]

**Lemma 10**

The smooth component \( v \) and singular component \( w \) and their derivatives satisfy the bounds for \( k=0,1,2,3 \)

\[
||v^{(k)}(x)|| \leq C(1 + \varepsilon^{2-k}), \quad \forall \ x \in \Omega_1 \cup \Omega_3 \text{ and}
\]

\[
|w^{(k)}(x)| \leq \begin{cases} 
C\varepsilon^{-k} e^{-\alpha(1+x)/\varepsilon}, \quad & \forall \ x \in \Omega_1 \\
C\varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}, \quad & \forall \ x \in \Omega_3
\end{cases}
\]

where \( |a(x)| \geq \alpha > 0, \quad \forall x \in \Omega_1 \cup \Omega_3. \)
Theorem 11

The smooth component $v$ and singular component $w$ and their derivatives satisfy the bounds for $k=0,1,2,3$

$$\| v^{(k)}(x) \| \leq C(1 + \varepsilon^{2-k}), \quad \text{and}$$

$$| w^{(k)}(x) | \leq C \varepsilon^{-k} (e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), \forall \ x \in \bar{\Omega}.$$
Finite Difference Scheme

- The problem (1)-(2) is discretized using classical finite difference scheme on piecewise uniform meshes (Shiskin mesh).
- The domain $\bar{\Omega}$ is divide into three subintervals $\Omega_L = [-1, -1 + \tau]$, $\Omega_C = [-1 + \tau, 1 - \tau]$ and $\Omega_R = [1 - \tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$.
- The transition parameter $\tau$ is chosen to be $\min\left\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\alpha}\right\}$.
- The domain $\bar{\Omega}^N$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both $\Omega_L$ and $\Omega_R$ and a uniform mesh with $N/2$ elements in $\Omega_C$. 

The resulting fitted finite difference scheme is to find $U(x_i)$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}^N$,

$$L_N^i U(x_i) := \varepsilon \delta^2 U(x_i) + a(x_i) D^* U(x_i) - b(x_i) U(x_i), \quad (8)$$

$$B_1^N U(x_0) = \beta_1 U(x_0) - \varepsilon \beta_2 D^+ U(x_0)$$

$$B_2^N U(x_N) = \gamma_1 U(x_N) + \varepsilon \gamma_2 D^- U(x_N), \quad (9)$$

where $D^+ U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}$, $D^- U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}$, $\delta^2 U(x_i) = \frac{D^+ U(x_i) - D^- U(x_i)}{(x_{i+1} - x_{i-1})/2}$ and

$$D^* U(x_i) = \begin{cases} 
D^+ U(x_i) & \text{if } a(x_i) > 0 \\
D^- U(x_i) & \text{if } a(x_i) < 0
\end{cases}.$$
Theorem 12 (Discrete minimum principle)

Let $L^N$ be the finite difference operator defined in (8)- (9) and let $\bar{\Omega}^N$ be an arbitrary mesh of $N + 1$ mesh points. If $\psi$ is any mesh function defined on this mesh such that $B_1^N \psi(x_0) \geq 0$, $B_2^N \psi(x_N) \geq 0$ and $L^N \psi(x_i) \leq 0$, for $i = 1(1)N - 1$ then

$$\psi(x_i) \geq 0, \quad \forall \ x_i \in \bar{\Omega}^N.$$ 

Lemma 13 (Discrete stability result)

Consider the scheme (8)- (9) to problem (1)-(3). If $\psi(x_i)$ is any mesh function then, for all $x_i \in \bar{\Omega}^N$

$$|\psi(x_i)| \leq C_{max}\{ |B_1^N \psi(x_0)|, |B_2^N \psi(x_N)|, \max_{1 \leq i \leq N-1} |L^N \psi(x_i)| \}. $$
Analogous to the continuous case, the discrete solution $U$ can be decomposed as

$$U = V + W,$$

where $V$ and $W$ are respectively the solutions of the problems

$$L^N V = f(x_i), \ x_i \in \bar{\Omega}^N,$$

$$\beta_1 V(-1) - \varepsilon \beta_2 D^+ V(-1) = \beta_1 v(-1) - \varepsilon \beta_2 v'(-1),$$
$$\gamma_1 V(1) + \varepsilon \gamma_2 D^- V(1) = \gamma_1 v(1) + \varepsilon \gamma_2 v'(1),$$

and

$$L^N W = 0, \ x_i \in \bar{\Omega}^N,$$

$$\beta_1 W(-1) - \varepsilon \beta_2 D^+ W(-1) = \beta_1 w(-1) - \varepsilon \beta_2 w'(-1),$$
$$\gamma_1 W(1) + \varepsilon \gamma_2 D^- W(1) = \gamma_1 w(1) + \varepsilon \gamma_2 w'(1).$$
Lemma 14

The error in the smooth component of the numerical solution is bounded as

\[ |(V - v)(x_i)| \leq CN^{-1}, \text{ for all } x_i \in \bar{\Omega}^N, \]

where \( v \) is the solution of (4) and \( V \) is the solution of (10).

Lemma 15

The error in the singular component of the numerical solution is bounded as

\[ |(W - w)(x_i)| \leq CN^{-1} \ln N, \forall x_i \in \bar{\Omega}^N, \]

where \( w \) is the solution of (7) and \( W \) is the solution of (11).

Theorem 16

If \( u \) is the solution of the problem (1) – (3) and \( U \) is the corresponding numerical solution using the method outlined in (8)-(9), then we have

\[ \sup_{0 < \varepsilon \leq 1} \| U - u \|_{\bar{\Omega}^N} \leq CN^{-1} \ln N \quad \forall \quad N \geq 4, \]
The following example is given to illustrate the numerical method. We use the double mesh principle given as in [5] to estimate the error and compute the experimental rate of convergence of the numerical method.

Define the double mesh differences to be

\[ D^N_\varepsilon = \left\{ \max_{x_i \in \bar{\Omega}^N} |U^N(x_i) - U^{2N}(x_i)| \right\}, \text{ and } D^N = \max_{\varepsilon} D^N_\varepsilon \]

where \( U^N(x_i) \) and \( U^{2N}(x_i) \) respectively, denote the numerical solution obtained using \( N \) and \( 2N \) mesh intervals. Further, we calculate the Robust order of convergence as

\[ p^N = \log_2 \left( \frac{D^N}{D^{2N}} \right). \]
The following example has a turning point at \( x = 1/2 \).

**Example 17**

\[
\varepsilon u''(x) - 2(2x - 1)u'(x) - 4u(x) = 4(4x - 15), \quad x \in (0, 1)
\]
\[
u(0) - \varepsilon u'(0) = 1, \quad u(1) + \varepsilon u'(1) = 1
\]

**Table:** Values of \( D^N \), \( p^N \) for the solution \( u \) for Example (17)

<table>
<thead>
<tr>
<th>Number of mesh points N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D^N )</td>
<td>9.2291e-1</td>
<td>5.6755e-1</td>
<td>3.7305e-1</td>
<td>2.2919e-1</td>
<td>1.3447e-1</td>
</tr>
<tr>
<td>( p^N )</td>
<td>7.0144e-1</td>
<td>6.0539e-1</td>
<td>7.0282e-1</td>
<td>7.6921e-1</td>
<td>-</td>
</tr>
</tbody>
</table>
Figure: Solution graph of Example 17 for various values of $\epsilon (\text{eps})$ and $N = 64$
Let the positive constants $\tilde{h}$ and $K$ be known. We construct a non uniform mesh on $\Omega_L$ as follows:

$$\tilde{h}_j = \tilde{h}_{j-1} + K \left[ \frac{\tilde{h}_{j-1}}{\varepsilon} \right] \min(\tilde{h}_{j-1}^2, \varepsilon), \quad j = 2(1)N/4.$$ 

Let $\tilde{q} = \sum_{j=1}^{N/4} \tilde{h}_j$, $q = \frac{\tau}{\tilde{q}}$, $h_j = q\tilde{h}_j$, $j = 1(1)N/4$.

An uniform mesh on $\Omega_c$, is defined as

$$h_j = \frac{4(1 - \tau)}{N}, \quad j = N/4 + 1(1)3N/4.$$ 

As in $\Omega_L$, a nonuniform mesh is constructed on $\Omega_R$ as $h_j = h_{N+1-j}$, $j = 3N/4 + 1(1)N$ and define $x_0 = -1$, $x_j = x_{j-1} + h_j$, $j = 1(1)N$. 
The cubic spline interpolating polynomial will have the following properties:

(i) \( S_j(x) \) coincides with the polynomial of degree three on each interval \([x_{j-1}, x_j], j = 1, 2, \cdots, N\)

(ii) \( S_j(x) \in C^2[0, 1] \),

(iii) \( S_j(x_j) = u(x_j), \ j = 0(1)N \).

Then we have the cubic spline functions,

\[
S_j(x) = \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + \left( u_{j-1} - \frac{h_j^2 M_{j-1}}{6} \right) \left( \frac{x_j - x}{h_j} \right) + \left( u_j - \frac{h_j^2 M_j}{6} \right) \left( \frac{x - x_{j-1}}{h_j} \right),
\]

where, \( x \in [x_{j-1}, x_j], \ h_j = x_j - x_{j-1}, \ j = 1(1)N \) and \( M_j = S_j''(x_j), \ j = 0(1)N \).
We obtain the difference scheme as

\[ L^N u_j = Qf_j, \quad j = 1(1)N - 1. \tag{12} \]

where,

\[
\begin{align*}
L^N u_j &= r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}, \\
Qf_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}
\end{align*}
\]

\[
\begin{align*}
 r_j^- &= \frac{2h_j + h_{j+1}}{6(h_j + h_{j+1})} a_{j-1} + \frac{h_{j+1}}{3h_j} a_j - \frac{h^2_{j+1}}{6h_j(h_j + h_{j+1})} a_{j+1} + \frac{h_j}{6} b_{j-1} - \frac{\varepsilon}{h_j}, \\
r_j^+ &= \frac{h^2_j}{6h_{j+1}(h_j + h_{j+1})} a_{j-1} - \frac{h_j}{3h_{j+1}} a_j - \frac{2h_{j+1} + h_j}{6(h_j + h_{j+1})} a_{j+1} + \frac{h_{j+1}}{6} b_{j+1} - \frac{\varepsilon}{h_{j+1}}, \\
r_j^c &= -\frac{h_j + h_{j+1}}{6h_{j+1}} a_{j-1} - \frac{h^2_{j+1} - h^2_j}{3h_jh_{j+1}} a_j + \frac{h_{j+1} + h_j}{6h_j} a_{j+1} + \frac{h_{j+1} + h_j}{3} b_j + \frac{\varepsilon}{h_j} + \frac{\varepsilon}{h_{j+1}}, \\
 q_j^- &= -\frac{h_j}{6}, \quad q_j^+ = -\frac{h_{j+1}}{6}, \quad q_j^c = -\frac{h_j + h_{j+1}}{3}
\end{align*}
\]
We approximate the first derivative by centered finite difference operator:

\[ B_1 u(x_0) \equiv \beta_1 u(x_0) - \varepsilon \beta_2 D^0 u(x_0) = A \quad \text{and} \quad B_2 u(x_N) \equiv \gamma_1 u(x_N) + \varepsilon \gamma_2 D^0 u(x_N) = B \]

That is,

\[ \beta_1 u_0 - \frac{\varepsilon \beta_2}{2h_0} [u_1 - u_{-1}] = A \quad \text{(or)} \quad u_{-1} = \frac{-2\beta_1 h_0}{\varepsilon \beta_2} u_0 + u_1 + \frac{2h_0 A}{\varepsilon \beta_2} \quad (13) \]

and

\[ \gamma_1 u_N + \frac{\varepsilon \gamma_2}{2h_N} [u_{N+1} - u_{N-1}] = B \quad \text{(or)} \quad u_{N+1} = \frac{-2\gamma_1 h_N}{\varepsilon \gamma_2} u_N + u_{N-1} + \frac{2h_N B}{\varepsilon \gamma_2} \quad (14) \]

where \( u(x_{-1}) \) and \( u(x_{N+1}) \) are the functional values at \( x_{-1} \) and \( x_{N+1} \).
The nodes $x_{-1}$ and $x_{N+1}$ lie outside the interval $[0, 1]$ and are called fictitious nodes. The values $u(x_{-1})$ and $u(x_{N+1})$ may be eliminated by assuming that the difference equation (12) holds also for $i = 0$ and $i = N$, that is at the boundary points $x_0$ and $x_N$. Substituting the values of $u_{-1}$ and $u_{N+1}$ from (13) and (14) into the equations (12) for $i = 0$ and $i = N$, we get respectively,

$$B_1^N \equiv \left[ \frac{r_0^c - 2\beta_1 h_0 r_0^-}{\varepsilon \beta_2} \right] u_0 + \left[ r_0^+ + r_0^- \right] u_1 = q_0^- f_{-1} + q_0^c f_0 + q_0^+ f_1 - \frac{2h_0 A r_0^-}{\varepsilon \beta_2} \quad (15)$$

and

$$B_2^N \equiv \left[ \frac{r_N^c - 2\gamma_1 h_N r_N^+}{\varepsilon \gamma_2} \right] u_N + \left[ r_N^- + r_N^+ \right] u_{N-1} = q_N^- f_{N-1} + q_N^c f_N + q_N^+ f_N + \frac{2h_N B r_N^+}{\varepsilon \gamma_2} \quad (16)$$
Theorem 18

Let \( \{ u_j \} \), \( j = 0(1)N \), be a set of values of the approximate solution to \( u(x) \) of (1)-(3), obtained by using (12), (15) and (16). Then there are positive constants \( C \) and \( \alpha \) (independent of \( h \) and \( \varepsilon \)) such that the following estimate holds:

\[
\max_j |u_j - u(x_j)| \leq Ch_c^2 \left[ \exp \left\{ \frac{-\alpha(1 + x_j)}{\varepsilon} \right\} + \exp \left\{ \frac{-\alpha(1 - x_j)}{\varepsilon} \right\} \right]
\]

where \( h_c = \max_j h_j \) = a constant.

Table: Values of \( D^N, p^N \) for the solution components \( u \) for the above Example (17)

<table>
<thead>
<tr>
<th>Number of mesh points N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D^N )</td>
<td>1.4089</td>
<td>3.0437e-1</td>
<td>7.3039e-2</td>
<td>1.7835e-2</td>
<td>5.2069e-3</td>
</tr>
<tr>
<td>( p^N )</td>
<td>2.2107</td>
<td>2.0591</td>
<td>2.0339</td>
<td>1.7762</td>
<td>-</td>
</tr>
</tbody>
</table>

J. Christy Roja (Department of Mathematics)
Find \( u_1, u_2 \in Y = C^0(\bar{\Omega}) \cap C^2(\Omega) \) such that

\[
\bar{L} \bar{u}(x) = \begin{cases}
L_1 \bar{u}(x) = \varepsilon \bar{u}_{1}''(x) + a_1(x)\bar{u}_1'(x) + b_{11}(x)u_1(x) \\
+ b_{12}(x)u_2(x) = f_1(x), \quad x \in \Omega,
L_2 \bar{u}(x) = \varepsilon \bar{u}_{2}''(x) + a_2(x)\bar{u}_2'(x) + b_{21}(x)u_1(x) \\
+ b_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega,
\end{cases}
\]

(17)

\[
u_1(-1) = l_1, \quad u_2(-1) = l_2, \quad u_1(1) = l_3, \quad u_2(1) = l_4
\]

(18)

\[
\left\{
\begin{array}{l}
b_{12} \geq 0, \quad b_{21} \geq 0, \quad b_{11} + b_{12} \leq 0, \quad b_{22} + b_{21} \leq 0 \\
|a_k(x)| \leq \alpha_k > 0, \quad \text{for} \quad 0 < |x| \leq 1, \quad a_k(0) = 0, \quad a_k'(0) < 0, \\
\text{and} \quad |a_k'(x)| \geq |a_k'(0)|/2 \quad \forall x \in \bar{\Omega}, \quad \text{for} \quad k = 1, 2
\end{array}
\right.
\]

(19)

where the functions \( a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x) \) and \( f_2(x) \) are sufficiently smooth on \( \bar{\Omega} \).
Finite difference scheme for the problem (17)-(18)

The fitted finite difference scheme is to find \( \bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T \) for \( i = 0, 1, 2, \cdots N \) such that for \( x_i \in \bar{\Omega}^N \),

\[
L_1^N \bar{U}(x_i) := \varepsilon \delta^2 U_1(x_i) + a_1(x_i) D^* U_1(x_i) + b_{11}(x_i) U_1(x_i) \tag{20}
+ b_{12}(x_i) U_2(x_i) = f_1(x_i) \quad i = 1(1)N - 1,
\]

\[
L_2^N \bar{U}(x_i) := \varepsilon \delta^2 U_2(x_i) + a_2(x_i) D^* U_2(x_i) + b_{21}(x_i) U_1(x_i) \tag{21}
+ b_{22}(x_i) U_2(x_i) = f_2(x_i), \quad i = 1(1)N - 1,
\]

\[
U_1(x_0) = u_1(-1), U_1(x_N) = u_1(1),
\]

\[
U_2(x_0) = u_2(-1), U_2(x_N) = u_2(1).
\]

where \( D^+ U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i} \), \( D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}} \),

\[
\delta^2 U_j(x_i) = \frac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}
\]

and

\[
D^* U_j(x_i) = \begin{cases} 
D^+ U_j(x_i) & \text{if } a_j(x_i) > 0 \\
D^- U_j(x_i) & \text{if } a_j(x_i) < 0
\end{cases}
\]
Theorem 19

Let $\tilde{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \bar{\Omega}$ be the solution of (17)-(19) and let $\tilde{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \bar{\Omega}^N$ be the numerical solution of problem (20)-(21). Then we have

$$\sup_{0<\varepsilon \leq 1} \| U_1 - u_1 \|_{\bar{\Omega}^N} \leq CN^{-1}(\ln N)^2 \quad \text{and} \quad \sup_{0<\varepsilon \leq 1} \| U_2 - u_2 \|_{\bar{\Omega}^N} \leq CN^{-1}(\ln N)^2.$$
Example 20

Consider the following system of singularly perturbed turning point problem

\[
\begin{align*}
\varepsilon u_1''(x) - 2(2x - 1)u_1'(x) - 9u_1'(x) + 2u_2(x) &= 0, \quad x \in (0, 1) \\
\varepsilon u_2''(x) - 4(2x - 1)u_2'(x) - 6u_2'(x) + u_1(x) &= 0, \quad x \in (0, 1) \\
u_1(0) &= 1, \quad u_2(0) = 1, \quad u_1(1) = 1, \quad u_2(1) = 1.
\end{align*}
\]
**Table:** Values of $D_1^N$, $p_1^N$ and $D_2^N$, $p_2^N$ for the solution components $U_1$ and $U_2$ respectively for Example 20

<table>
<thead>
<tr>
<th></th>
<th>Number of mesh points N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>64</td>
</tr>
<tr>
<td>$D_1^N$</td>
<td>2.3351e-2</td>
</tr>
<tr>
<td>$p_1^N$</td>
<td>6.8307e-1</td>
</tr>
<tr>
<td>$p_2^N$</td>
<td>9.4103e-1</td>
</tr>
</tbody>
</table>
Figure: Solution graph of Example 20 for $\varepsilon = 2^{-4}$ and $N = 2^7$
Figure: Maximum pointwise errors as a function of $N$ and $\varepsilon$ for the solution $U_1$ and $U_2$ for Example 20
Problem class III: Weakly coupled system of second order SPTPPs with Robin boundary conditions

Find \( \tilde{u} = (u_1, u_2)^T \in Y = C^1(\overline{\Omega}) \cap C^2(\Omega) \) such that

\[
\bar{L}\tilde{u}(x) = \begin{cases} 
L_1\tilde{u}(x) = \varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) \\
+ b_{12}(x)u_2(x) = f_1(x), \quad x \in \Omega,
\end{cases} \\
L_2\tilde{u}(x) = \varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) \\
+ b_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega,
\]

with the boundary conditions

\[
\begin{align*}
B_{10}u_1(-1) &\equiv \beta_{10}u_1(-1) - \varepsilon \beta_{11}u_1'(-1) = A_1, \\
B_{11}u_1(1) &\equiv \gamma_{11}u_1(1) + \varepsilon \gamma_{12}u_1'(1) = B_1, \\
B_{20}u_2(-1) &\equiv \beta_{20}u_2(-1) - \varepsilon \beta_{21}u_2'(-1) = A_2, \\
B_{21}u_2(1) &\equiv \gamma_{21}u_2(1) + \varepsilon \gamma_{22}u_2'(1) = B_2,
\end{align*}
\]
and the assumptions

\[
\begin{align*}
&b_{12} \geq 0, \ b_{21} \geq 0, \ b_{11} + b_{12} \leq b_1 < 0, \ b_{22} + b_{21} \leq b_2 < 0 \\
|a_k(x)| \leq \alpha_k > 0, \ \text{for} \ 0 < |x| \leq 1, \ a_k(0) = 0, \ \dot{a}_k'(0) < 0, \\
&\alpha_k + b_k < 0 \ \text{and} \ \left|\dot{a}_k'(x)\right| \geq \left|\dot{a}_k'(0)\right|/2 \ \forall x \in \bar{\Omega}, \ \text{for} \ k = 1, 2 \\
&\beta_{j0}, \beta_{j1} \geq 0, \ \beta_{j0} - \varepsilon \beta_{j1} \geq 0, \ \gamma_{j1}, \gamma_{j2} \geq 0, \ j = 1, 2.
\end{align*}
\]

(24)

where the functions \(a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x)\) and \(f_2(x)\) are sufficiently smooth on \(\bar{\Omega}\),
Finite difference scheme for the problem (22)-(23)

The fitted finite difference scheme is to find \( \overline{U}(x_i) = (U_1(x_i), U_2(x_i))^T \) for \( i = 0, 1, 2, \cdots N \) such that for \( x_i \in \overline{\Omega}^N \),

\[
\begin{align*}
\overline{L}^N U(x_i) &= \begin{cases} 
  L_1^N \overline{U}(x_i) := \varepsilon \delta^2 U_1(x_i) + a_1(x_i) D^* U_1(x_i) + b_{11}(x_i) U_1(x_i) \\
  + b_{12}(x_i) U_2(x_i) = f_1(x_i), & i = 1(1)N - 1, \\
  L_2^N \overline{U}(x_i) := \varepsilon \delta^2 U_2(x_i) + a_2(x_i) D^* U_2(x_i) + b_{21}(x_i) U_1(x_i) \\
  + b_{22}(x_i) U_2(x_i) = f_2(x_i), & i = 1(1)N - 1,
\end{cases} 
\end{align*}
\]

\[
\begin{align*}
B_{10}^N U_1(x_0) &\equiv \beta_{10} U_1(x_0) - \varepsilon \beta_{11} D^+ U_1(x_0) = A_1, \\
B_{11}^N U_1(x_N) &\equiv \gamma_{11} U_1(x_N) + \varepsilon \gamma_{12} D^- U_1(x_N) = B_1, \\
B_{20}^N U_2(x_0) &\equiv \beta_{20} U_2(x_0) - \varepsilon \beta_{21} D^+ U_2(x_0) = A_2, \\
B_{21}^N U_2(x_N) &\equiv \gamma_{21} U_2(x_N) + \varepsilon \gamma_{22} D^- U_2(x_N) = B_2,
\end{align*}
\]
Theorem 21

Let \( \bar{u}(x) = (u_1(x), u_2(x))^T \), for all \( x \in \bar{\Omega} \) be the solution of (22)-(24) and let \( \bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T \), for all \( x_i \in \bar{\Omega}^N \) be the numerical solution of problem (25)-(26). Then we have

\[
\sup_{0 < \varepsilon \leq 1} \| U_1 - u_1 \|_{\bar{\Omega}_\varepsilon^N} \leq C N^{-1} \ln N \quad \text{and} \quad \sup_{0 < \varepsilon \leq 1} \| U_2 - u_2 \|_{\bar{\Omega}^N} \leq C N^{-1} \ln N.
\]
Example 22

Consider the following system of singularly perturbed turning point problem

\[ \varepsilon u^{\prime\prime}_1(x) - 7(2x - 1)u^{\prime}_1(x) - 10u_1(x) + 2u_2(x) = -e^x, \quad x \in (0, 1) \]
\[ \varepsilon u^{\prime\prime}_2(x) - 3(2x - 1)u^{\prime}_2(x) - 7u_2(x) + 3u_1(x) = x + 5, \quad x \in (0, 1) \]
\[ u_1(0) - \varepsilon u^{\prime}_1(0) = 2, \quad u_2(0) - \varepsilon u^{\prime}_2(0) = 2, \]
\[ u_1(1) + \varepsilon u^{\prime}_1(1) = 2, \quad u_2(1) + \varepsilon u^{\prime}_2(1) = 2. \]
Table: Values of $D_1^N$, $p_1^N$ and $D_2^N$, $p_2^N$ for the solution components $U_1$ and $U_2$ respectively for Example 22

<table>
<thead>
<tr>
<th>Number of mesh points N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1^N$</td>
<td>5.3386e-2</td>
<td>3.3316e-2</td>
<td>2.0917e-2</td>
<td>1.2528e-2</td>
<td>6.3892e-3</td>
</tr>
<tr>
<td>$p_1^N$</td>
<td>6.8027e-1</td>
<td>6.7153e-1</td>
<td>7.3946e-1</td>
<td>9.7149e-1</td>
<td>-</td>
</tr>
<tr>
<td>$D_2^N$</td>
<td>5.7580e-2</td>
<td>3.4862e-2</td>
<td>2.0375e-2</td>
<td>1.1785e-2</td>
<td>6.5330e-3</td>
</tr>
<tr>
<td>$p_2^N$</td>
<td>7.2390e-1</td>
<td>7.7486e-1</td>
<td>7.8991e-1</td>
<td>8.5108e-1</td>
<td>-</td>
</tr>
</tbody>
</table>
Figure: Solution graph of Example 22 for $\varepsilon = 2^{-4}$ and $N = 2^7$
Figure: Maximum pointwise errors as a function of $N$ and $\varepsilon$ for the solution $u_1$ and $u_2$ for Example 22
Find $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ such that

$$
\begin{cases}
Lu = \varepsilon u'''(x) + a(x)u''(x) - b(x)u'(x) + c(x)u(x) = f(x), x \in \Omega, \\
u(-1) = l_1, \ u'(-1) = l_2, \ u'(1) = l_3,
\end{cases}
$$

(27)

$$
\begin{cases}
|a(x)| \leq \alpha > 0, \text{ for } 0 < |x| \leq 1, \ a(0) = 0, \ a'(0) < 0, \\
\beta_0 \geq b(x) \geq \beta_0 > 0, \ \gamma_0 \geq c(x) \geq \gamma_0 > 0, \ \alpha < \beta_0 - \gamma^0, \\
\text{and } |a'(x)| \geq |a'(0)|/2 \ \forall x \in \bar{\Omega},
\end{cases}
$$

(28)

where $a(x), b(x), c(x)$ and $f(x)$ are smooth functions on $\bar{\Omega}$.
The above problem is equivalent to the following problem:

Find $\bar{u} = (u_1, u_2)^T$, $u_1, u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\bar{L}\bar{u} = \begin{cases} L_1\bar{u} := u_2(x) - u_1'(x) = 0, & x \in \Omega, \\ L_2\bar{u} := \varepsilon u_2''(x) + a(x)u_2'(x) - b(x)u_2(x) + c(x)u_1(x) = f(x), & x \in \Omega, \end{cases}$$

(29)

$$u_1(-1) = l_1, \quad u_2(-1) = l_2, \quad u_2(1) = l_3.$$  \hspace{1cm} (30)
The fitted finite difference scheme is to find \( \tilde{U}(x_i) = (U_1(x_i), U_2(x_i))^T \) for \( i = 0, 1, 2, \cdots N \) such that for \( x_i \in \tilde{\Omega}^N \),

\[
L_1^N \tilde{U}(x_i) := U_2(x_i) - D^- U_1(x_i) = 0, \quad i = 1(1)N, \\
L_2^N \tilde{U}(x_i) := \varepsilon \delta^2 U_2(x_i) + a(x_i) D^* U_2(x_i) - b(x_i) U_2(x_i) + c(x_i) U_1(x_i) = f(x_i), \quad i = 1(1)N - 1,
\]

\[
U_1(x_0) = u_1(x_0), \quad U_2(x_0) = u_2(x_0), \quad U_2(x_N) = u_2(x_N).
\]

where \( D^+ U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i} \), \( D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}} \),

\[
\delta^2 U_j(x_i) = \frac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}
\]

and

\[
D^* U_j(x_i) = \begin{cases} 
D^+ U_j(x_i) & \text{if } a(x_i) > 0 \\
D^- U_j(x_i) & \text{if } a(x_i) < 0
\end{cases}
\]
Theorem 23

Let $\tilde{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \tilde{\Omega}$ be the solution of (29)-(30) and let $\tilde{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \tilde{\Omega}_\varepsilon^N$ be the numerical solution of problem (31). Then we have

$$\sup_{0<\varepsilon\leq 1} \| U_1 - u_1 \|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N \quad \text{and} \quad \sup_{0<\varepsilon\leq 1} \| U_2 - u_2 \|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N.$$
Example 24

Consider the following singularly perturbed turning point problem

\[ \varepsilon u'''(x) - 5xu''(x) - (x + 4)u'(x) + (2 + x)u(x) = -e^x, \quad x \in (-1, 1) \]

\[ u(-1) = 1, \quad u'(-1) = 1, \quad u'(1) = 1. \]

**Table:** Values of \( D_1^N, p_1^N \) and \( D_2^N, p_2^N \) for the solution components \( U_1 \) and \( U_2 \) respectively for Example 24

<table>
<thead>
<tr>
<th>Number of mesh points N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1^N )</td>
<td>2.7087e-2</td>
<td>1.5076e-2</td>
<td>8.0262e-3</td>
<td>4.0350e-3</td>
<td>2.0241e-3</td>
</tr>
<tr>
<td>( p_1^N )</td>
<td>8.4539e-1</td>
<td>9.0943e-1</td>
<td>9.9214e-1</td>
<td>9.9530e-1</td>
<td>-</td>
</tr>
<tr>
<td>( D_2^N )</td>
<td>6.7643e-2</td>
<td>4.5055e-2</td>
<td>2.5059e-2</td>
<td>1.2807e-2</td>
<td>7.7298e-3</td>
</tr>
<tr>
<td>( p_2^N )</td>
<td>5.8625e-1</td>
<td>8.4638e-1</td>
<td>9.6833e-1</td>
<td>7.2848e-1</td>
<td>-</td>
</tr>
</tbody>
</table>
**Figure:** Solution graph of Example 24 for $\varepsilon = 2^{-4}$ and $N = 2^7$
Figure: Maximum pointwise errors as a function of $N$ and $\epsilon$ for the solution $u_1$ and $u_2$ for Example 24
Find \( u \in C^2(\bar{\Omega}) \cap C^4(\Omega) \) such that

\[
Lu = -\varepsilon u^{iv}(x) - a(x)u'''(x) + b(x)u''(x) + c(x)u(x) = f(x), \quad x \in \Omega,
\]

\[
\begin{align*}
u(-1) &= l_1, \quad u(1) = l_2, \quad u''(-1) = l_3, \quad u''(1) = l_4,
\end{align*}
\] (32)

with the assumptions

\[
\begin{align*}
|a(x)| &\leq \alpha > 0, \quad \text{for } 0 < |x| \leq 1, \quad a(0) = 0,
\end{align*}
\]

\[
\begin{align*}
&\alpha < \beta_0 - \gamma^0, \quad \text{and } |a'(x)| \geq |a'(0)|/2 \quad \forall x \in \bar{\Omega},
\end{align*}
\] (33)

where \( a(x), b(x), c(x) \) and \( f(x) \) are smooth functions on \( \bar{\Omega} \).
The above problem is equivalent to the following problem:

Find \( \bar{u} = (u_1, u_2)^T \), \( u_1, u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega) \) such that

\[
\bar{L}\bar{u} = \begin{cases} 
L_1 \bar{u} := u_1''(x) + u_2(x) = 0, & x \in \Omega, \\
L_2 \bar{u} := \varepsilon u_2''(x) + a(x)u_2'(x) - b(x)u_2(x) + c(x)u_1(x) = f(x), & x \in \Omega,
\end{cases}
\]

(34)

\[u_1(-1) = l_1, \quad u_1(1) = l_2, \quad u_2(-1) = l_3, \quad u_2(1) = l_4.\]

(35)
Finite difference scheme for the problem (34)-(35)

The fitted finite difference scheme is to find \( \bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T \) for \( i = 0, 1, 2, \cdots N \) such that for \( x_i \in \bar{\Omega}^N_\epsilon \), is \( \bar{L}^N = (L^N_1, L^N_2) \) where

\[
L^N_1 \bar{U}(x_i) := \delta^2 U_1(x_i) + U_2(x_i) = 0, \quad i = 1(1)N - 1, \\
L^N_2 \bar{U}(x_i) := \epsilon \delta^2 U_2(x_i) + a(x_i) D^* U_2(x_i) - b(x_i) U_2(x_i) + c(x_i) U_1(x_i) = f(x_i), \quad i = 1(1)N - 1, \\
U_1(x_0) = u_1(x_0), \quad U_1(x_N) = u_1(x_N), \\
U_2(x_0) = u_2(x_0), \quad U_2(x_N) = u_2(x_N),
\]

where \( D^+ U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i} \), \( D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}} \),

\[
\delta^2 U_j(x_i) = \frac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2} \quad \text{and} \\
D^* U_j(x_i) = \begin{cases} 
  D^+ U_j(x_i) & \text{if } a(x_i) > 0 \\
  D^- U_j(x_i) & \text{if } a(x_i) < 0
\end{cases}
\]
Theorem 25

Let $\tilde{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \tilde{\Omega}$, be the solution of (34)-(35) and let $\tilde{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \tilde{\Omega}^N$, be the numerical solution of problem (36). Then we have

$$\sup_{0 < \varepsilon \leq 1} \left\| U_1 - u_1 \right\|_{\tilde{\Omega}^N} \leq CN^{-1} \ln N \quad \text{and} \quad \sup_{0 < \varepsilon \leq 1} \left\| U_2 - u_2 \right\|_{\tilde{\Omega}^N} \leq CN^{-1} \ln N$$
Example 26

Consider the following singularly perturbed turning point problem

\[-\varepsilon u^{(4)}(x) + 5xu^{(3)}(x) + (4 + x)u^{(2)}(x) + (2 + x)u(x) = -\exp(x), \quad x \in (-1, 1)\]

\[u(-1) = 1, \quad u(1) = 1, \quad u^{(2)}(-1) = 1, \quad u^{(2)}(1) = 1.\]
**Table:** Values of $D_1^N$, $p_1^N$ and $D_2^N$, $p_2^N$ for the solution components $U_1$ and $U_2$ respectively for Example 26

<table>
<thead>
<tr>
<th>Number of mesh points N</th>
<th>$D_1^N$</th>
<th>$D_2^N$</th>
<th>$p_1^N$</th>
<th>$p_2^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>3.9254e-2</td>
<td>3.5852e-2</td>
<td>9.7697e-1</td>
<td>4.9207e-1</td>
</tr>
<tr>
<td>128</td>
<td>1.9943e-2</td>
<td>2.5491e-2</td>
<td>9.8861e-1</td>
<td>8.5727e-1</td>
</tr>
<tr>
<td>256</td>
<td>1.0050e-2</td>
<td>1.4071e-2</td>
<td>9.9433e-1</td>
<td>8.0104e-1</td>
</tr>
<tr>
<td>512</td>
<td>5.0450e-3</td>
<td>8.0757e-3</td>
<td>9.9717e-1</td>
<td>8.1256e-1</td>
</tr>
<tr>
<td>1024</td>
<td>2.5274e-3</td>
<td>4.5981e-3</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
**Figure:** Solution graph of Example 26 for $\varepsilon = 2^{-4}$ and $N = 2^7$
Figure: Maximum pointwise errors as a function of $N$ and $\varepsilon$ for the solution $u_1$ and $u_2$ for Example 26
Conclusion and Scope

- Finite Difference method and Variable mesh spline approximation method are applied for problem class I. Finite Difference method is used to solve the remaining problems.

- One can apply these methods for other classes of problems like multiple turning point problems, turning point problem with interior layers, two parameter turning point problems, turning point problem with discontinuous source term, etc.


1 N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for second order singularly perturbed turning point problems with Robin boundary conditions.*

Problems working on


- N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for singularly perturbed turning point problems with discontinuous source term.*


Thank You