Numerical methods for singularly perturbed differential equations with turning points

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Plan of talk

- The birth of the SPPs was introduced by Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904 and it was reported in the proceedings of the conference.
- Many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems are described by differential equations involving large or small parameters.
- The solutions of SPPs have non-uniform behavior. That is, there are thin layer(s) (boundary layer region) where the solution varies rapidly while away from the layer(s) (outer region) the solution behaves regularly and varies slowly.

Let P_ε denote the original problem and u_ε be its solution.
Let P_0 denote the reduced problem of P_ε (setting $\varepsilon = 0$ in P_ε) and u_0 be its solution.
Then the problem P_ε is called a **Singular Perturbation Problem(SPP)** if and only if u_ε does not converge uniformly to u_0 in the entire domain of the definition of the problem. Otherwise the problem is called **Regular Perturbation Problem (RPP)**.

Example 1 (Regular Perturbation Problem)

$$P_\varepsilon : \quad \begin{cases} u'_\varepsilon(x) = -\varepsilon u_\varepsilon(x),\ x \in (0,1], \\ u_\varepsilon(0) = 1, \quad 0 < \varepsilon \ll 1. \end{cases}$$

$$P_0 : \quad u'_0(x) = 0,\ x \in (0,1], u_0(0) = 1,$$

Example 2 (Singular Perturbation Problem)

$$P_\varepsilon : \quad \begin{cases} \varepsilon u_\varepsilon'(x) = -u_\varepsilon(x), \ x \in (0,1], \\ u_\varepsilon(0) = 1, \quad 0 < \varepsilon \ll 1. \end{cases}$$

$$P_0 : \quad u_0(x) = 0, \ x \in [0,1],$$

The exact solution is given by

$$u_\varepsilon(x) = \exp(-x/\varepsilon).$$

Note that,

$$\lim_{\varepsilon \to 0} \lim_{x \to 0} u_\varepsilon(x) = 1,$$

$$\lim_{x \to 0} \lim_{\varepsilon \to 0} u_\varepsilon(x) = 0.$$

That is, $u_\varepsilon(x)$ does not converge uniformly to the reduced problem solution on $[0,1]$.

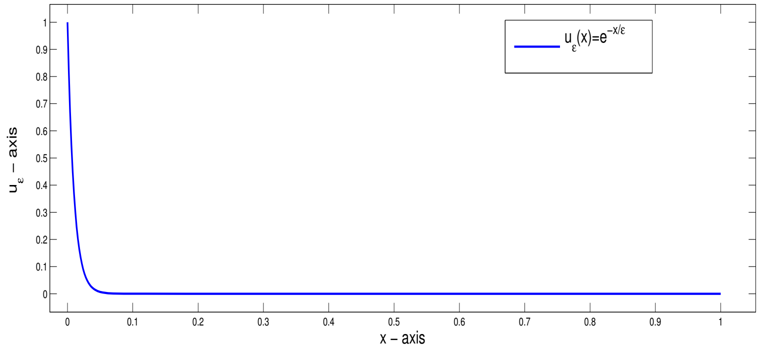

Figure: Graph of the Solution for $\varepsilon = 0.01$

The solution changes very rapidly near the neighborhood of $x = 0$. This neighborhood is called a **boundary layer**.

- Often these mathematical problems are extremely difficult (or even impossible) to solve exactly and in these circumstances approximate solutions are necessary. One can obtain an approximate solution through the use of perturbation methods.
- In general, regular numerical methods like Euler method, Runge Kutta methods, finite difference methods, etc cannot be applied to these SPPs.

For Example

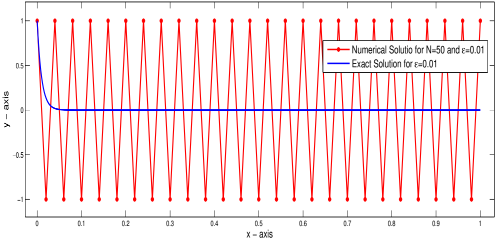

Figure: Euler Method for Example (2)

- The disadvantage in the classical numerical methods (finite difference/ finite element) is due to the nature of the coefficients. That is, inaccurate solution due to perturbation parameter.
- Classical numerical methods on equidistant grids yield satisfactory numerical solution for singularly perturbed boundary value problems only if one uses an unacceptably large number of grid points.
- In order to overcome this difficulty we apply numerical methods on appropriate meshes like Shishkin mesh, Bakhvalov mesh, Bakhvalov Shishkin mesh etc.

Introduction to Singularly Perturbed Turning Point Problems (SPTPPs)

- The main difference between singular perturbation problem and singularly perturbed turning point problem is the coefficient of the convection term vanishes inside the domain of the differential equation.
- If the turning point occur at the interior of the domain, then the problem is called as an interior turning point problem, otherwise it is a boundary turning point problem.
- If the velocity distribution is linear, then the problem is known as a simple turning point problem, otherwise it is a multiple turning point problem.

Mathematical model for a turning point problem

Consider the one dimensional equation [3] which describes a quantum mechanical particle in a potential $V(x)$

$$\left(-\varepsilon^2 \frac{d^2}{dx^2} + V(x) - E\right) y(x) = 0,$$

where $V(x)$ is the potential energy of the particle and
E is the total energy of the particle

- For this equation , $Q(x) = V(x) - E$, so $Q(x)$ vanishes at points where $V(x) = E$ and these are called turning points.
- The classical orbit of a particle in the potential $V(x)$ is confined to the regions where $V(x) \leq E$.
- The particle moves until it reaches a point where $V = E$ and then it stops, turns around and moves of in the opposite direction.

Applications

SPTPPs occur in the modelling of following problems.

- Modeling of steady and unsteady viscous flow problems with large Reynolds number
- Navier Stokes flows with large Reynolds numbers
- Magneto-hydrodynamic duct problems at high Hartman numbers
- Heat transport problem with large Peclet numbers
- One dimensional version of stationary convection-diffusion problems with a dominant convective term
- Speed field that changes its sign in the catch basin
- Geophysics and modeling thermal boundary layers in laminar flow.

A typical linear turning point problem in one dimension[12] is given by

Example 3

$$-\varepsilon u^{''}(x) + xb(x)u^{'}(x) + c(x)u(x) = f(x),\ x \in (-1, 1),\ u(-1) = u(1) = 0$$

under the following assumptions:

$(i)\ \ b(x) \neq 0$ on $[-1, 1]\ \ (ii)\ \ c(x) \geq 0,\ c(0) > 0.$

- The location of any boundary layer(s) depends on the sign of the convection term.
- From our experience, we expect a boundary layer at $x = -1$ if the coefficient of the convection term $xb(x)$ is negative at $x = -1$, and a boundary layer at $x = 1$ if the same coefficient is positive at $x = 1$.
- If $b(x)$ is positive on $[-1, 1]$, we have $xb(x)|_{x=-1} < 0$ and $xb(x)|_{x=1} > 0$.
- Consequently, if b is positive on $[-1, 1]$, then the solution u has two boundary layers at $x = 1$ and $x = -1$ otherwise the solution has interior layer at $x = 0$.

Example 4 (Exhibiting layers at the boundary)

Consider the BVP

$$\begin{aligned} &\varepsilon u''(x) - 2(2x-1)u'(x) - 4u(x) = 0 \;\; \forall \;\; x \in (0,1) \\ &u(0) = 1, \;\; u(1) = 1 \end{aligned}$$

The exact solution is given by

$$u(x) = e^{-2x(1-x)/\varepsilon}$$

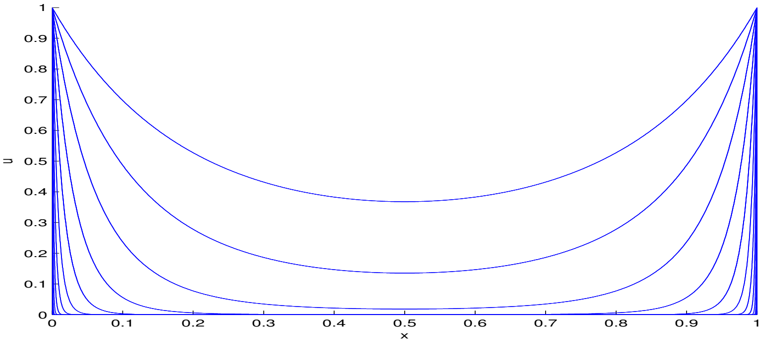

Figure: Exact solution of example 4 for $\varepsilon = 2^{-2}$ to $\varepsilon = 2^{-10}$ and $N = 1024$

Example 5 (Exhibiting layers at the interior)

Consider the BVP

$$\varepsilon u''(x) + 2xu'(x) = 0 \;\; \forall \;\; x \in (-1, 1)$$
$$u(-1) = -1, \;\; u(1) = 1$$

The exact solution is given by

$$u(x) = erf(x/\sqrt{\varepsilon})$$

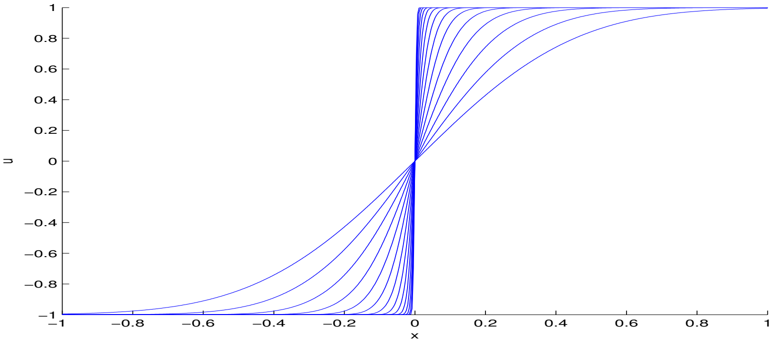

Figure: Exact solution of example 5 for $\varepsilon = 2^{-2}$ to $\varepsilon = 2^{-10}$ and $N = 1024$

- In the present thesis, motivated by the works of [1, 4, 6, 9, 11, 16, 17, 19, 20, 21], two methods are given namely
  - **Parameter Uniform Finite Difference Method(PUFDM)**
  - **Variable Mesh Spline Approximation Method(VMSAM)**

  for various singularly perturbed turning point problems.
- The PUFDM and VMSAM are discussed for Problem class I, where as the PUFDM is applied for Problem classes II to V.

Problem class I: Second order SPTPPs with Robin boundary conditions

Find $u \in C^1(\bar{\Omega} = [-1, 1]) \cap C^2(\Omega = (-1, 1))$ such that

$$Lu \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \ \ \forall \ \ x \in \Omega \tag{1}$$

with Robin boundary conditions

$$B_1 u(-1) = \beta_1 u(-1) - \varepsilon\beta_2 u'(-1) = A \tag{2}$$
$$B_2 u(1) = \gamma_1 u(1) + \varepsilon\gamma_2 u'(1) = B$$

and the asssumptions

$$\begin{cases} a(0) = 0, \ \ a'(0) < 0, |a(x)| \leq \alpha_0 > 0, 0 < \beta_0 \leq b(x), \\ \alpha_0 < \beta_0, |a'(x)| \geq \dfrac{|a'(0)|}{2} \forall x \in \bar{\Omega}, \beta_1, \beta_2 \geq 0, \\ \beta_1 - \varepsilon\beta_2 > 0, \ \gamma_2 \geq 0 \ \& \ \gamma_1 > 0. \end{cases} \tag{3}$$

where $\varepsilon (0 < \varepsilon \ll 1)$ is a small positive parameter, $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions on $\bar{\Omega}$.

Theorem 6 (Minimum Principle)

Let L be the differential operator defined in (1) and $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. If $B_1 v(-1) \geq 0, \;\; B_2 v(1) \geq 0$ and $Lv \leq 0 \;\; \forall \;\; x \in \Omega$, then $v(x) \geq 0 \;\;\; \forall \;\; x \in \bar{\Omega}$.

Lemma 7 (Stability Result)

If $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, then

$$|u(x)| \leq C \max \left\{ \max\{|B_1 u|, |B_2 u|\}, ||Lu||_{x \in \Omega} \right\}, \forall x \in \bar{\Omega}.$$

From now on we shall denote the subdomains of $\bar{\Omega} = [-1, 1]$ as $\Omega_1 = [-1, -\delta]$, $\Omega_2 = [-\delta, \delta]$ and $\Omega_3 = [\delta, 1]$, $0 < \delta \leq 1/2$. The choice of $\delta = 1/2$ can be found in [4].

Lemma 8

Let u be the solution of (1)-(3). Then

$$\begin{aligned} ||u^{(k)}|| &\leq C\varepsilon^{-(k)} \max\{||f||, ||u||\}, \quad k = 1, 2 \\ ||u^{(3)}|| &\leq C\varepsilon^{-(3)} \max\{||f||, ||f'||, ||u||\}, \end{aligned}$$

$\forall x \in \Omega_1 \cup \Omega_3$, *where C depends on* $||a||, ||a'||, ||b||$ *and* $||b'||$.

The following lemma gives estimates for u and its derivatives in the interval Ω_2 which includes the turning point $x = 0$.

Lemma 9

Let u be the solution of (1)-(3). Then

$$||u^{(k)}(x)|| \leq C, \quad \forall x \in \Omega_2,$$

where C depends on $||a||, ||a^{'}||, ||b||, ||b^{'}||, ||f||, ||f^{'}||$ *and* β.

To derive ε - uniform error estimates we require sharper bounds of the solution and its derivatives. For this we use Shishkin decomposition of the solution u as

$$u = v + w.$$

Here v is the solution of the problem

$$\begin{aligned} &Lv = f \qquad (4)\\ &\beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1) = \beta_1 v_0(-1) - \varepsilon\beta_2 v_0^{'}(-1) + \varepsilon(\beta_1 v_1(-1) - \varepsilon\beta_2 v_1^{'}(-1)),\\ &\gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1) = \gamma_1 v_0(1) + \varepsilon\gamma_2 v_0^{'}(1) + \varepsilon(\gamma_1 v_1(1) + \varepsilon\gamma_2 v_1^{'}(1)) \end{aligned}$$

where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$.
Also v_0 and v_1 are defined respectively, to be the solutions of the reduced problem:

$$av_0^{'} - bv_0 = f \text{ and } av_1^{'} - bv_1 = -v_0^{''} \qquad (5)$$

and v_2 is the solution of the problem similar to that defining u

$$\begin{aligned} &Lv_2 = -v_1^{''}, \qquad (6)\\ &\beta_1 v_2(-1) - \varepsilon\beta_2 v_2^{'}(-1) = 0, \;\; \gamma_1 v_2(1) + \varepsilon\gamma_2 v_2^{'}(1) = 0. \end{aligned}$$

The singular component w is the solution of the homogeneous problem

$$
\begin{aligned}
&Lw = 0, && (7)\\
&\beta_1 w(-1) - \varepsilon\beta_2 w^{'}(-1) = (\beta_1 u(-1) - \varepsilon\beta_2 u^{'}(-1)) - (\beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1)),\\
&\gamma_1 w(1) + \varepsilon\gamma_2 w^{'}(1) = (\gamma_1 u(1) + \varepsilon\gamma_2 u^{'}(1)) - (\gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1)).
\end{aligned}
$$

Lemma 10

The smooth component v and singular component w and their derivatives satisfy the bounds for k=0,1,2,3

$$
\begin{aligned}
||v^{(k)}(x)|| &\leq C(1+\varepsilon^{2-k}), \ \forall \ x \in \Omega_1 \cup \Omega_3 \ \textit{and}\\
|w^{(k)}(x)| &\leq \begin{cases} C\varepsilon^{-k} e^{-\alpha(1+x)/\varepsilon}, & \forall \ x \in \Omega_1 \\ C\varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}, & \forall \ x \in \Omega_3 \end{cases}
\end{aligned}
$$

where $|a(x)| \geq \alpha > 0, \ \forall x \in \Omega_1 \cup \Omega_3$.

Theorem 11

The smooth component v and singular component w and their derivatives satisfy the bounds for k=0,1,2,3

$$
\begin{aligned}
||v^{(k)}(x)|| &\leq C(1+\varepsilon^{2-k}), \quad \textit{and} \\
|w^{(k)}(x)| &\leq C\varepsilon^{-k}(e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), \;\; \forall\, x \in \bar{\Omega}.
\end{aligned}
$$

Finite Difference Scheme

- The problem (1)-(2) is discretized using classical finite difference scheme on piecewise uniform meshes (Shiskin mesh).
- The domain $\bar{\Omega}$ is divide into three subintervals $\Omega_L = [-1, -1+\tau]$, $\Omega_C = [-1+\tau, 1-\tau]$ and $\Omega_R = [1-\tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$.
- The transition parameter τ is chosen to be $\min\left\{\frac{1}{2}, \frac{2\varepsilon lnN}{\alpha}\right\}$.
- The domain $\bar{\Omega}^N$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both Ω_L and Ω_R and a uniform mesh with $N/2$ elements in Ω_C.

The resulting fitted finite difference scheme is to find $U(x_i)$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}^N$,

$$L^N U(x_i) := \varepsilon\delta^2 U(x_i) + a(x_i) D^* U(x_i) - b(x_i) U(x_i), \tag{8}$$

$$B_1^N U(x_0) = \beta_1 U(x_0) - \varepsilon\beta_2 D^+ U(x_0)$$

$$B_2^N U(x_N) = \gamma_1 U(x_N) + \varepsilon\gamma_2 D^- U(x_N), \tag{9}$$

where $D^+ U(x_i) = \dfrac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}$, $D^- U(x_i) = \dfrac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}$,

$\delta^2 U(x_i) = \dfrac{D^+ U(x_i) - D^- U(x_i)}{(x_{i+1} - x_{i-1})/2}$ and

$$D^* U(x_i) = \begin{cases} D^+ U(x_i) & \text{if} \quad a(x_i) > 0 \\ D^- U(x_i) & \text{if} \quad a(x_i) < 0 \end{cases} .$$

Theorem 12 (Discrete minimum principle)

Let L^N be the finite difference operator defined in (8)- (9) and let $\bar{\Omega}^N$ be an arbitrary mesh of $N+1$ mesh points. If ψ is any mesh function defined on this mesh such that
$B_1^N\psi(x_0) \geq 0,\ B_2^N\psi(x_N) \geq 0$ and $L^N\psi(x_i) \leq 0,$ for $i = 1(1)N-1$ then

$$\psi(x_i) \geq 0, \ \forall\, x_i \in \bar{\Omega}^N.$$

Lemma 13 (Discrete stability result)

Consider the scheme (8)- (9) to problem (1)-(3). If $\psi(x_i)$ is any mesh function then, for all $x_i \in \bar{\Omega}^N$

$$|\psi(x_i)| \leq C max\{|B_1^N\psi(x_0)|, |B_2^N\psi(x_N)|, \max_{1\leq i\leq N-1} |L^N\psi(x_i)|\}.$$

Analogous to the continuous case, the discrete solution U can be decomposed as

$$U = V + W,$$

where V and W are respectively the solutions of the problems

$$\begin{aligned} &L^N V = f(x_i), \quad x_i \in \bar{\Omega}^N, \\ &\beta_1 V(-1) - \varepsilon\beta_2 D^+ V(-1) = \beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1), \\ &\gamma_1 V(1) + \varepsilon\gamma_2 D^- V(1) = \gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &L^N W = 0, \quad x_i \in \bar{\Omega}^N, \\ &\beta_1 W(-1) - \varepsilon\beta_2 D^+ W(-1) = \beta_1 w(-1) - \varepsilon\beta_2 w^{'}(-1), \\ &\gamma_1 W(1) + \varepsilon\gamma_2 D^- W(1) = \gamma_1 w(1) + \varepsilon\gamma_2 w^{'}(1). \end{aligned} \tag{11}$$

Lemma 14

The error in the smooth component of the numerical solution is bounded as

$$|(V - v)(x_i)| \leq CN^{-1}, \text{ for all } x_i \in \bar{\Omega}^N,$$

where v is the solution of (4) and V is the solution of (10).

Lemma 15

The error in the singular component of the numerical solution is bounded as

$$|(W - w)(x_i)| \leq CN^{-1} lnN, \forall x_i \in \bar{\Omega}^N,$$

where w is the solution of (7) and W is the solution of (11).

Theorem 16

If u is the solution of the problem $(1) - (3)$ *and U is the corresponding numerical solution using the method outlined in (8)-(9), then we have*

$$\sup_{0<\varepsilon\leq 1} ||U - u||_{\bar{\Omega}^N} \leq CN^{-1} lnN \ \forall \ N \geq 4,$$

The following example is given to illustrate the numerical method. We use the double mesh principle given as in [5] to estimate the error and compute the experimental rate of convergence of the numerical method.
Define the double mesh differences to be

$$D_\varepsilon^N = \left\{ \max_{x_i \in \bar{\Omega}^N} |U^N(x_i) - U^{2N}(x_i)| \right\}, \text{ and } D^N = \max_{\varepsilon} D_\varepsilon^N$$

where $U^N(x_i)$ and $U^{2N}(x_i)$ respectively, denote the numerical solution obtained using N and $2N$ mesh intervals. Further, we calculate the Robust order of convergence as

$$p^N = log_2\left(\frac{D^N}{D^{2N}}\right).$$

The following example has a turning point at $x = 1/2$.

Example 17

$$\varepsilon u^{''}(x) - 2(2x-1)u^{'}(x) - 4u(x) = 4(4x-15), \qquad x \in (0,1)$$

$$u(0) - \varepsilon u^{'}(0) = 1, \;\; u(1) + \varepsilon u^{'}(1) = 1$$

Table: *Values of D^N, p^N for the solution u for Example (17)*

	Number of mesh points N				
	64	128	256	512	1024
D^N	9.2291e-1	5.6755e-1	3.7305e-1	2.2919e-1	1.3447e-1
p^N	7.0144e-1	6.0539e-1	7.0282e-1	7.6921e-1	-

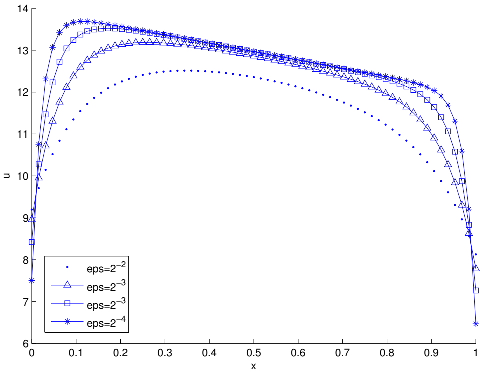

Figure: Solution graph of Example 17 for various values of ε(*eps*) and $N = 64$

Let the positive constants $\tilde{h}$ and K be known.
We construct a non uniform mesh on Ω_L as follows:

$$\tilde{h}_j = \tilde{h}_{j-1} + K\left[\frac{\tilde{h}_{j-1}}{\varepsilon}\right]\min(\tilde{h}^2_{j-1}, \varepsilon), \quad j = 2(1)N/4.$$

$$\text{Let} \quad \tilde{q} = \sum_{j=1}^{N/4} \tilde{h}_j, \quad q = \frac{\tau}{\tilde{q}}, \quad h_j = q\tilde{h}_j, \quad j = 1(1)N/4$$

An uniform mesh on Ω_c, is defined as

$$h_j = \frac{4(1-\tau)}{N}, \quad j = N/4 + 1\ (1)\ 3N/4.$$

As in Ω_L, a nonuniform mesh is constructed on Ω_R as
$h_j = h_{N+1-j}, \quad j = 3N/4 + 1(1)N$
and define $x_0 = -1, \quad x_j = x_{j-1} + h_j, \quad j = 1(1)N.$

The cubic spline interpolating polynomial will have the following properties:

(i) $S_j(x)$ coincides with the polynomial of degree three on each interval $[x_{j-1}, x_j]$, $j = 1, 2, \cdots, N$

(ii) $S_j(x) \in \mathbb{C}^2[0, 1]$,

(iii) $S_j(x_j) = u(x_j), \;\; j = 0(1)N.$

Then we have the cubic spline functions,

$$
\begin{aligned}
S_j(x) \;\; &= \;\; \frac{(x_j - x)^3}{6h_j} M_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} M_j + \left(u_{j-1} - \frac{h_j^2 M_{j-1}}{6}\right)\left(\frac{x_j - x}{h_j}\right) + \\
&\left(u_j - \frac{h_j^2 M_j}{6}\right)\left(\frac{x - x_{j-1}}{h_j}\right),
\end{aligned}
$$

where, $x \in [x_{j-1}, x_j],\; h_j = x_j - x_{j-1},\; j = 1(1)N$ and $M_j = S_j^{''}(x_j),\; j = 0(1)N.$

We obtain the difference scheme as

$$L^N u_j = Qf_j, \quad j = 1(1)N-1. \tag{12}$$

where, $L^N u_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}, \quad Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$

$$r_j^- = \frac{2h_j + h_{j+1}}{6(h_j + h_{j+1})} a_{j-1} + \frac{h_{j+1}}{3h_j} a_j - \frac{h_{j+1}^2}{6h_j(h_j + h_{j+1})} a_{j+1} + \frac{h_j}{6} b_{j-1} - \frac{\varepsilon}{h_j},$$

$$r_j^+ = \frac{h_j^2}{6h_{j+1}(h_j + h_{j+1})} a_{j-1} - \frac{h_j}{3h_{j+1}} a_j - \frac{2h_{j+1} + h_j}{6(h_j + h_{j+1})} a_{j+1} + \frac{h_{j+1}}{6} b_{j+1} - \frac{\varepsilon}{h_{j+1}},$$

$$r_j^c = -\frac{h_j + h_{j+1}}{6h_{j+1}} a_{j-1} - \frac{h_{j+1}^2 - h_j^2}{3h_j h_{j+1}} a_j + \frac{h_{j+1} + h_j}{6h_j} a_{j+1} + \frac{h_{j+1} + h_j}{3} b_j + \frac{\varepsilon}{h_j} + \frac{\varepsilon}{h_{j+1}},$$

$$q_j^- = -\frac{h_j}{6}, \quad q_j^+ = -\frac{h_{j+1}}{6}, \quad q_j^c = -\frac{h_j + h_{j+1}}{3}$$

We approximate the first derivative by centered finite difference operator:
$B_1 u(x_0) \equiv \beta_1 u(x_0) - \varepsilon\beta_2 D^0 u(x_0) = A$ and
$B_2 u(x_N) \equiv \gamma_1 u(x_N) + \varepsilon\gamma_2 D^0 u(x_N) = B$
That is,

$$\beta_1 u_0 - \frac{\varepsilon\beta_2}{2h_0}[u_1 - u_{-1}] = A \text{ (or) } u_{-1} = \frac{-2\beta_1 h_0}{\varepsilon\beta_2}u_0 + u_1 + \frac{2h_0 A}{\varepsilon\beta_2} \quad (13)$$

and

$$\gamma_1 u_N + \frac{\varepsilon\gamma_2}{2h_N}[u_{N+1} - u_{N-1}] = B \text{ (or) } u_{N+1} = \frac{-2\gamma_1 h_N}{\varepsilon\gamma_2}u_N + u_{N-1} + \frac{2h_N B}{\varepsilon\gamma_2} \quad (14)$$

where $u(x_{-1})$ and $u(x_{N+1})$ are the functional values at x_{-1} and x_{N+1}.

The nodes x_{-1} and x_{N+1} lie outside the interval $[0,1]$ and are called fictitious nodes. The values $u(x_{-1})$ and $u(x_{N+1})$ may be eliminated by assuming that the difference equation (12) holds also for $i = 0$ and $i = N$, that is at the boundary points x_0 and x_N. Substituting the values of u_{-1} and u_{N+1} from (13) and (14) into the equations (12) for $i = 0$ and $i = N$, we get respectively,

$$B_1^N \equiv \left[r_0^c - \frac{2\beta_1 h_0 r_0^-}{\varepsilon\beta_2}\right] u_0 + [r_0^+ + r_0^-]u_1 = q_0^- f_{-1} + q_0^c f_0 + q_0^+ f_1 - \frac{2h_0 A r_0^-}{\varepsilon\beta_2} \quad (15)$$

and

$$B_2^N \equiv \left[r_N^c - \frac{2\gamma_1 h_N r_N^+}{\varepsilon\gamma_2}\right] u_N + [r_N^- + r_N^+]u_{N-1} = q_N^- f_{N-1} + q_N^c f_N + q_N^+ f_{N+1} - \frac{2h_N B r_N^+}{\varepsilon\gamma_2}. \quad (16)$$

Theorem 18

Let $\{u_j\},\ j = 0(1)N$, be a set of values of the approximate solution to $u(x)$ of (1)-(3), obtained by using (12), (15) and (16). Then there are positive constants C and α (independent of h and ε) such that the following estimate holds:

$$\max_j |u_j - u(x_j)| \leq Ch_c^2 \left[exp\left\{ \frac{-\alpha(1+x_j)}{\varepsilon} \right\} + exp\left\{ \frac{-\alpha(1-x_j)}{\varepsilon} \right\} \right]$$

where $h_c = \max_j h_j = a\ \ constant$.

Table: *Values of D^N, p^N for the solution components u for the above Example (17)*

	Number of mesh points N				
	64	128	256	512	1024
D^N	1.4089	3.0437e-1	7.3039e-2	1.7835e-2	5.2069e-3
p^N	2.2107	2.0591	2.0339	1.7762	-

Find $u_1, u_2 \in Y = C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\bar{L}\bar{u}(x) = \begin{cases} L_1\bar{u}(x) = \varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) \\ \qquad +b_{12}(x)u_2(x) = f_1(x), x \in \Omega, \\ L_2\bar{u}(x) = \varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) \\ \qquad +b_{22}(x)u_2(x) = f_2(x), x \in \Omega, \end{cases} \tag{17}$$

$$u_1(-1) = l_1, \;\; u_2(-1) = l_2, \;\; u_1(1) = l_3, \;\; u_2(1) = l_4 \tag{18}$$

$$\begin{cases} b_{12} \geq 0, \;\; b_{21} \geq 0, \;\; b_{11} + b_{12} \leq 0, \;\; b_{22} + b_{21} \leq 0 \\ |a_k(x)| \leq \alpha_k > 0, \quad \text{for} \quad 0 < |x| \leq 1, \;\; a_k(0) = 0, \;\; a_k'(0) < 0, \\ \quad \text{and } |a_k'(x)| \geq |a_k'(0)|/2 \;\; \forall x \in \bar{\Omega}, \; \text{for } k = 1, 2 \end{cases} \tag{19}$$

where the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x)$ and $f_2(x)$ are sufficiently smooth on $\bar{\Omega}$.

Finite difference scheme for the problem (17)-(18)

The fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}^N$,

$$\begin{aligned} L_1^N \bar{U}(x_i) &:= \varepsilon\delta^2 U_1(x_i) + a_1(x_i) D^* U_1(x_i) + b_{11}(x_i) U_1(x_i) \qquad (20) \\ &\quad + b_{12}(x_i) U_2(x_i) = f_1(x_i) \qquad i = 1(1)N-1, \\ L_2^N \bar{U}(x_i) &:= \varepsilon\delta^2 U_2(x_i) + a_2(x_i) D^* U_2(x_i) + b_{21}(x_i) U_1(x_i) \qquad (21) \\ &\quad + b_{22}(x_i) U_2(x_i) = f_2(x_i), \quad i = 1(1)N-1, \\ &\quad U_1(x_0) = u_1(-1), U_1(x_N) = u_1(1), \\ &\quad U_2(x_0) = u_2(-1), U_2(x_N) = u_2(1). \end{aligned}$$

where $D^+ U_j(x_i) = \dfrac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}$, $D^- U_j(x_i) = \dfrac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}$,

$\delta^2 U_j(x_i) = \dfrac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}$ and

$$D^* U_j(x_i) = \begin{cases} D^+ U_j(x_i) & \text{if} \quad a_j(x_i) > 0 \\ D^- U_j(x_i) & \text{if} \quad a_j(x_i) < 0 \end{cases}.$$

Theorem 19

Let $\bar{u}(x) = (u_1(x), u_2(x))^T$*, for all* $x \in \bar{\Omega}$ *be the solution of (17)-(19)and let* $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$*, for all* $x_i \in \bar{\Omega}^N$ *be the numerical solution of problem (20)-(21). Then we have*

$$\sup_{0<\varepsilon\leq 1} ||U_1 - u_1||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}(lnN)^2 \text{ and } \sup_{0<\varepsilon\leq 1} ||U_2 - u_2||_{\bar{\Omega}^N} \leq CN^{-1}(lnN)^2.$$

Example 20

Consider the following system of singularly perturbed turning point problem

$$\varepsilon u_1''(x) - 2(2x-1)u_1'(x) - 9u_1'(x) + 2u_2(x) = 0, \quad x \in (0,1)$$
$$\varepsilon u_2''(x) - 4(2x-1)u_2'(x) - 6u_2'(x) + u_1(x) = 0, \quad x \in (0,1)$$
$$u_1(0) = 1, \quad u_2(0) = 1, \quad u_1(1) = 1, \quad u_2(1) = 1.$$

Table: *Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components U_1 and U_2 respectively for Example 20*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	2.3351e-2	1.4544e-2	8.9904e-3	5.2098e-3	2.9596e-3
p_1^N	6.8307e-1	6.9394e-1	7.8715e-1	8.1584e-1	-
D_2^N	8.0236e-2	4.1792e-2	2.1304e-2	1.1884e-2	6.1696e-3
p_2^N	9.4103e-1	9.7207e-1	8.4213e-1	9.4575e-1	-

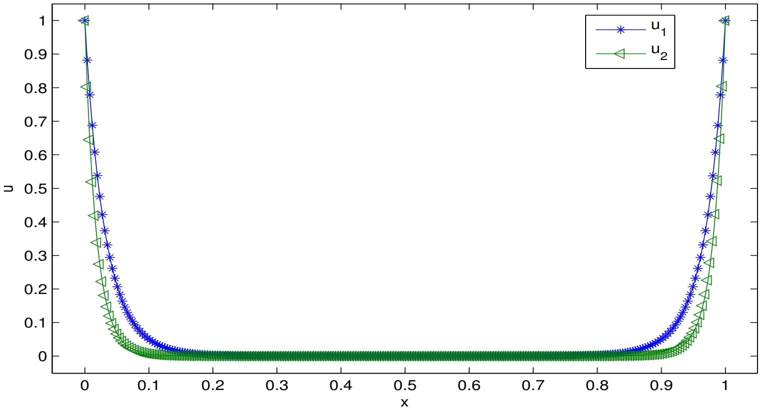

Figure: Solution graph of Example 20 for $\varepsilon = 2^{-4}$ and $N = 2^7$

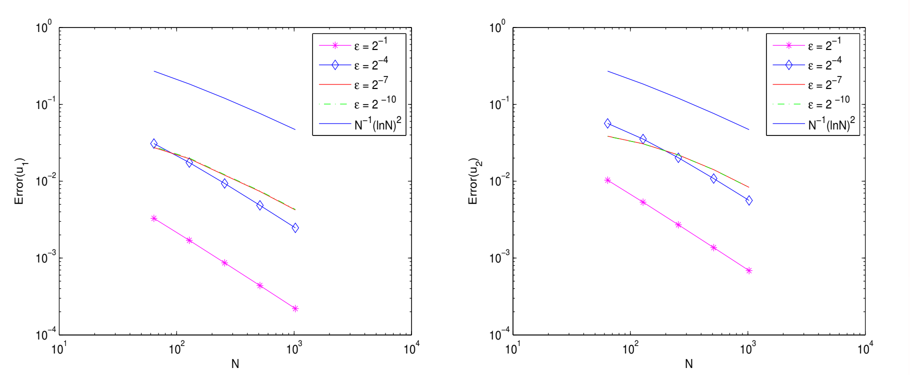

Figure: Maximum pointwise errors as a function of N and ε for the solution U_1 and U_2 for Example 20

Problem class III: Weakly coupled system of second order SPTPPs with Robin boundary conditions

Find $\bar{u} = (u_1, u_2)^T \in Y = C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\bar{L}\bar{u}(x) = \begin{cases} L_1\bar{u}(x) = \varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) \\ \qquad +b_{12}(x)u_2(x) = f_1(x), \quad x \in \Omega, \\ L_2\bar{u}(x) = \varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) \\ \qquad +b_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega, \end{cases} \tag{22}$$

with the boundary conditions

$$\begin{cases} B_{10}u_1(-1) \equiv \beta_{10}u_1(-1) - \varepsilon\beta_{11}u_1'(-1) = A_1, \\ B_{11}u_1(1) \equiv \gamma_{11}u_1(1) + \varepsilon\gamma_{12}u_1'(1) = B_1, \\ B_{20}u_2(-1) \equiv \beta_{20}u_2(-1) - \varepsilon\beta_{21}u_2'(-1) = A_2, \\ B_{21}u_2(1) \equiv \gamma_{21}u_2(1) + \varepsilon\gamma_{22}u_2'(1) = B_2, \end{cases} \tag{23}$$

and the assumptions

$$\begin{cases} b_{12} \geq 0, \;\; b_{21} \geq 0, \;\; b_{11} + b_{12} \leq b_1 < 0, \;\; b_{22} + b_{21} \leq b_2 < 0 \\ |a_k(x)| \leq \alpha_k > 0, \quad \text{for} \quad 0 < |x| \leq 1, \;\; a_k(0) = 0, \;\; a_k'(0) < 0, \\ \;\; \alpha_k + b_k < 0 \text{ and } |a_k'(x)| \geq |a_k'(0)|/2 \;\; \forall x \in \bar{\Omega}, \; \text{for } k = 1, 2 \\ \beta_{j0}, \beta_{j1} \geq 0, \beta_{j0} - \varepsilon\beta_{j1} \geq 0, \gamma_{j1}, \gamma_{j2} \geq 0, j = 1, 2. \end{cases} \tag{24}$$

where the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x)$, $f_1(x)$ and $f_2(x)$ are sufficiently smooth on $\bar{\Omega}$, The fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}^N$,

$$\bar{L}^N \bar{U}(x_i) = \begin{cases} L_1^N \bar{U}(x_i) := \varepsilon\delta^2 U_1(x_i) + a_1(x_i) D^* U_1(x_i) + b_{11}(x_i) U_1(x_i) \\ \qquad + b_{12}(x_i) U_2(x_i) = f_1(x_i), \quad i = 1(1)N-1, \\ L_2^N \bar{U}(x_i) := \varepsilon\delta^2 U_2(x_i) + a_2(x_i) D^* U_2(x_i) + b_{21}(x_i) U_1(x_i) \\ \qquad + b_{22}(x_i) U_2(x_i) = f_2(x_i), \quad i = 1(1)N-1, \end{cases} \tag{25}$$

$$\begin{cases} B_{10}^N U_1(x_0) \equiv \beta_{10} U_1(x_0) - \varepsilon\beta_{11} D^+ U_1(x_0) = A_1, \\ B_{11}^N U_1(x_N) \equiv \gamma_{11} U_1(x_N) + \varepsilon\gamma_{12} D^- U_1(x_N) = B_1, \\ B_{20}^N U_2(x_0) \equiv \beta_{20} U_2(x_0) - \varepsilon\beta_{21} D^+ U_2(x_0) = A_2, \\ B_{21}^N U_2(x_N) \equiv \gamma_{21} U_2(x_N) + \varepsilon\gamma_{22} D^- U_2(x_N) = B_2, \end{cases} \tag{26}$$

Theorem 21

Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \bar{\Omega}$ be the solution of (22)-(24)and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \bar{\Omega}^N$ be the numerical solution of problem (25)-(26). Then we have

$$\sup_{0<\varepsilon\leq 1} ||U_1 - u_1||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N \text{ and } \sup_{0<\varepsilon\leq 1} ||U_2 - u_2||_{\bar{\Omega}^N} \leq CN^{-1}\ln N.$$

Example 22

Consider the following system of singularly perturbed turning point problem

$$\begin{aligned}
&\varepsilon u_1''(x) - 7(2x-1)u_1'(x) - 10u_1(x) + 2u_2(x) = -e^x, \quad x \in (0,1)\\
&\varepsilon u_2''(x) - 3(2x-1)u_2'(x) - 7u_2(x) + 3u_1(x) = x+5, \quad x \in (0,1)\\
&u_1(0) - \varepsilon u_1'(0) = 2, \quad u_2(0) - \varepsilon u_2'(0) = 2,\\
&u_1(1) + \varepsilon u_1'(1) = 2, \quad u_2(1) + \varepsilon u_2'(1) = 2.
\end{aligned}$$

Table: *Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components U_1 and U_2 respectively for Example 22*

Number of mesh points N					
	64	128	256	512	1024
D_1^N	5.3386e-2	3.3316e-2	2.0917e-2	1.2528e-2	6.3892e-3
p_1^N	6.8027e-1	6.7153e-1	7.3946e-1	9.7149e-1	-
D_2^N	5.7580e-2	3.4862e-2	2.0375e-2	1.1785e-2	6.5330e-3
p_2^N	7.2390e-1	7.7486e-1	7.8991e-1	8.5108e-1	-

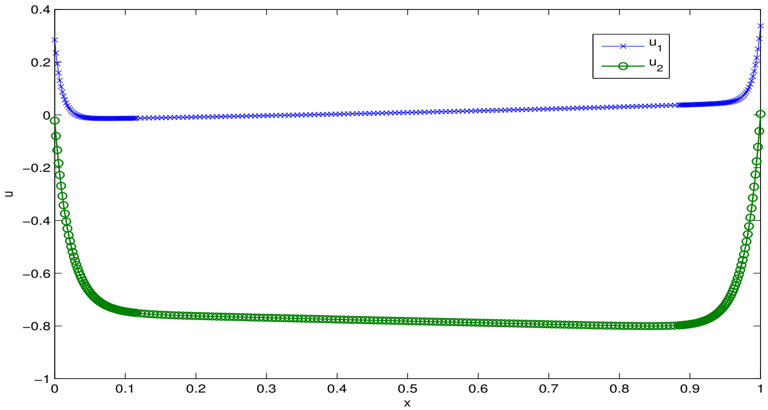

Figure: Solution graph of Example 22 for $\varepsilon = 2^{-4}$ and $N = 2^7$

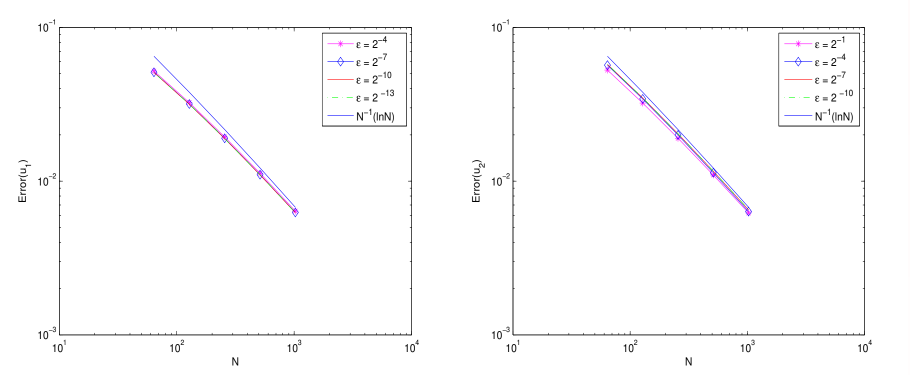

Figure: Maximum pointwise errors as a function of N and ε for the solution u_1 and u_2 for Example 22

Find $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ such that

$$\begin{cases} Lu = \varepsilon u'''(x) + a(x)u''(x) - b(x)u'(x) + c(x)u(x) = f(x), x \in \Omega, \\ u(-1) = l_1, \;\; u'(-1) = l_2, \;\; u'(1) = l_3, \end{cases} \tag{27}$$

$$\begin{cases} |a(x)| \leq \alpha > 0, \quad \text{for} \quad 0 < |x| \leq 1, \;\; a(0) = 0, \quad a'(0) < 0, \\ \beta^0 \geq b(x) \geq \beta_0 > 0, \;\; \gamma^0 \geq c(x) \geq \gamma_0 > 0, \;\; \alpha < \beta_0 - \gamma^0, \\ \text{and } |a'(x)| \geq |a'(0)|/2 \;\; \forall x \in \bar{\Omega}, \end{cases} \tag{28}$$

where $a(x), b(x), c(x)$ and $f(x)$ are smooth functions on $\bar{\Omega}$.

The above problem is equivalent to the following problem:

Find $\bar{u} = (u_1, u_2)^T$, $u_1, u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\bar{L}\bar{u} = \begin{cases} L_1\bar{u} := u_2(x) - u_1^{'}(x) = 0, \ x \in \Omega, \\ L_2\bar{u} := \varepsilon u_2^{''}(x) + a(x)u_2'(x) - b(x)u_2(x) \\ \qquad\qquad +c(x)u_1(x) = f(x), \ x \in \Omega, \end{cases} \tag{29}$$

$$u_1(-1) = l_1, \ \ u_2(-1) = l_2, \ \ u_2(1) = l_3. \tag{30}$$

The fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}^N$,

$$\begin{aligned} L_1^N \bar{U}(x_i) &:= U_2(x_i) - D^- U_1(x_i) = 0, \qquad i = 1(1)N, \\ L_2^N \bar{U}(x_i) &:= \varepsilon \delta^2 U_2(x_i) + a(x_i) D^* U_2(x_i) - b(x_i) U_2(x_i) \\ &\quad + c(x_i) U_1(x_i) = f(x_i), \qquad i = 1(1)N-1, \\ U_1(x_0) &= u_1(x_0), \quad U_2(x_0) = u_2(x_0), \quad U_2(x_N) = u_2(x_N). \end{aligned} \tag{31}$$

where $D^+ U_j(x_i) = \dfrac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}$, $D^- U_j(x_i) = \dfrac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}$,
$\delta^2 U_j(x_i) = \dfrac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}$ and
$D^* U_j(x_i) = \begin{cases} D^+ U_j(x_i) & \text{if} \quad a(x_i) > 0 \\ D^- U_j(x_i) & \text{if} \quad a(x_i) < 0 \end{cases}$.

Theorem 23

Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \bar{\Omega}$ be the solution of (29)-(30)and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \bar{\Omega}_\varepsilon^N$ be the numerical solution of problem (31). Then we have

$$\sup_{0<\varepsilon\leq 1} ||U_1 - u_1||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N \text{ and } \sup_{0<\varepsilon\leq 1} ||U_2 - u_2||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N.$$

Example 24

Consider the following singularly perturbed turning point problem

$$\varepsilon u^{'''}(x) - 5xu^{''}(x) - (x+4)u^{'}(x) + (2+x)u(x) = -e^{x}, \qquad x \in (-1,1)$$

$$u(-1) = 1, \quad u^{'}(-1) = 1, \quad u^{'}(1) = 1.$$

Table: *Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components U_1 and U_2 respectively for Example 24*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	2.7087e-2	1.5076e-2	8.0262e-3	4.0350e-3	2.0241e-3
p_1^N	8.4539e-1	9.0943e-1	9.9214e-1	9.9530e-1	-
D_2^N	6.7643e-2	4.5055e-2	2.5059e-2	1.2807e-2	7.7298e-3
p_2^N	5.8625e-1	8.4638e-1	9.6833e-1	7.2848e-1	-

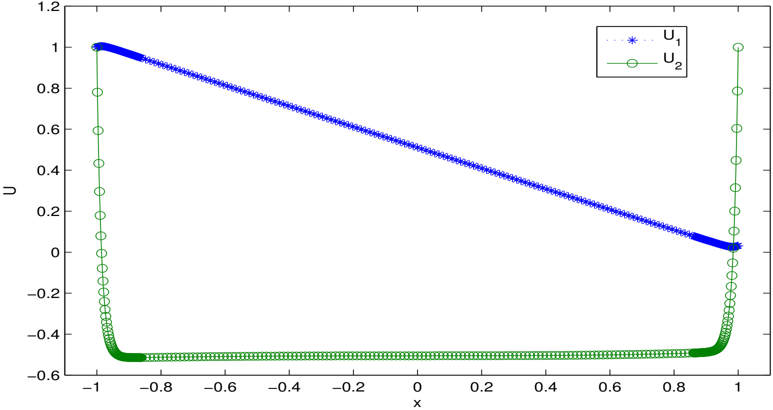

Figure: Solution graph of Example 24 for $\varepsilon = 2^{-4}$ and $N = 2^7$

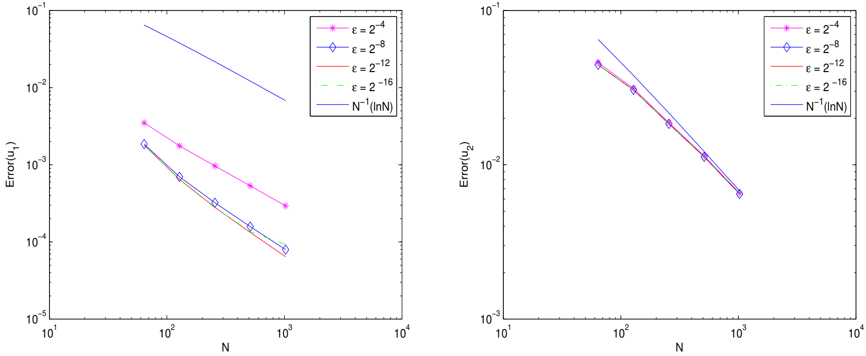

Figure: Maximum pointwise errors as a function of N and ε for the solution u_1 and u_2 for Example 24

Find $u \in C^2(\bar{\Omega}) \cap C^4(\Omega)$ such that

$$\begin{cases} Lu = -\varepsilon u^{iv}(x) - a(x)u'''(x) + b(x)u''(x) \\ \qquad\qquad\qquad +c(x)u(x) = f(x), x \in \Omega, \\ u(-1) = l_1, \;\; u(1) = l_2, \;\; u''(-1) = l_3, \;\; u''(1) = l_4, \end{cases} \tag{32}$$

with the assumptions

$$\begin{cases} |a(x)| \leq \alpha > 0, \;\; \text{for} \;\; 0 < |x| \leq 1, \; a(0) = 0, \\ a'(0) < 0, \; \beta^0 \geq b(x) \geq \beta_0 > 0, \; \gamma^0 \geq c(x) \geq \gamma_0 > 0, \\ \alpha < \beta_0 - \gamma^0, \; \text{and} \; |a'(x)| \geq |a'(0)|/2 \; \forall x \in \bar{\Omega}, \end{cases} \tag{33}$$

where $a(x), b(x), c(x)$ and $f(x)$ are smooth functions on $\bar{\Omega}$.

The above problem is equivalent to the following problem:

Find $\bar{u} = (u_1, u_2)^T, \quad u_1, u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\bar{L}\bar{u} = \begin{cases} L_1\bar{u} := u_1''(x) + u_2(x) = 0, \ x \in \Omega, \\ L_2\bar{u} := \varepsilon u_2''(x) + a(x)u_2'(x) - b(x)u_2(x) \\ \qquad\qquad +c(x)u_1(x) = f(x), \ x \in \Omega, \end{cases} \tag{34}$$

$$u_1(-1) = l_1, \ \ u_1(1) = l_2, \ \ u_2(-1) = l_3, \ \ u_2(1) = l_4. \tag{35}$$

The fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}_\varepsilon^N$, is $\bar{L}^N = (L_1^N, L_2^N)$ where

$$\begin{aligned} L_1^N \bar{U}(x_i) &:= \delta^2 U_1(x_i) + U_2(x_i) = 0, \qquad i = 1(1)N-1, \\ L_2^N \bar{U}(x_i) &:= \varepsilon \delta^2 U_2(x_i) + a(x_i) D^* U_2(x_i) - b(x_i) U_2(x_i) \\ &\quad + c(x_i) U_1(x_i) = f(x_i), \qquad i = 1(1)N-1, \\ &\quad U_1(x_0) = u_1(x_0), \quad U_1(x_N) = u_1(x_N), \\ &\quad U_2(x_0) = u_2(x_0), \quad U_2(x_N) = u_2(x_N), \end{aligned} \tag{36}$$

where $D^+ U_j(x_i) = \dfrac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}$, $D^- U_j(x_i) = \dfrac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}$,
$\delta^2 U_j(x_i) = \dfrac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}$ and

$$D^* U_j(x_i) = \begin{cases} D^+ U_j(x_i) & \text{if } \; a(x_i) > 0 \\ D^- U_j(x_i) & \text{if } \; a(x_i) < 0 \end{cases}.$$

Theorem 25

Let $\bar{u}(x) = (u_1(x), u_2(x))^T$*, for all* $x \in \bar{\Omega}$ *be the solution of (34)-(35)and let* $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$*, for all* $x_i \in \bar{\Omega}^N$ *be the numerical solution of problem (36). Then we have*

$$\sup_{0<\varepsilon\leq 1} ||U_1 - u_1||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}\ln N \text{ and } \sup_{0<\varepsilon\leq 1} ||U_2 - u_2||_{\bar{\Omega}^N} \leq CN^{-1}\ln N$$

Example 26

Consider the following singularly perturbed turning point problem

$$-\varepsilon u^{iv}(x)+5xu^{'''}(x)+(4+x)u^{''}(x)+(2+x)u(x)=-exp(x), \quad x\in(-1,1)$$

$$u(-1)=1, u(1)=1, u^{''}(-1)=1,\ u^{''}(1)=1.$$

Table: *Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components U_1 and U_2 respectively for Example 26*

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	3.9254e-2	1.9943e-2	1.0050e-2	5.0450e-3	2.5274e-3
p_1^N	9.7697e-1	9.8861e-1	9.9433e-1	9.9717e-1	-
D_2^N	3.5852e-2	2.5491e-2	1.4071e-2	8.0757e-3	4.5981e-3
p_2^N	4.9207e-1	8.5727e-1	8.0104e-1	8.1256e-1	-

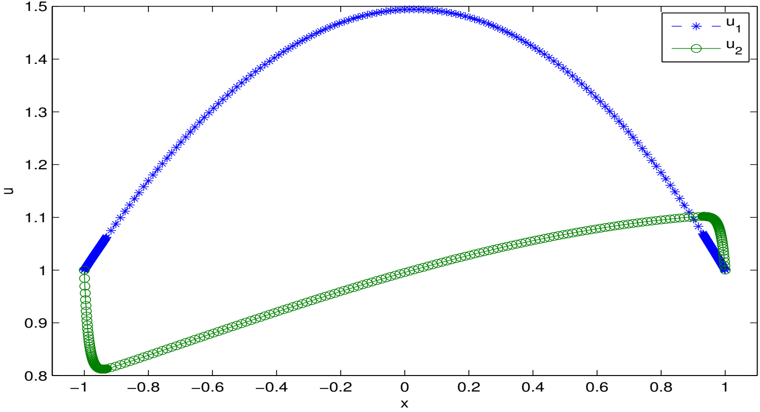

Figure: Solution graph of Example 26 for $\varepsilon = 2^{-4}$ and $N = 2^7$

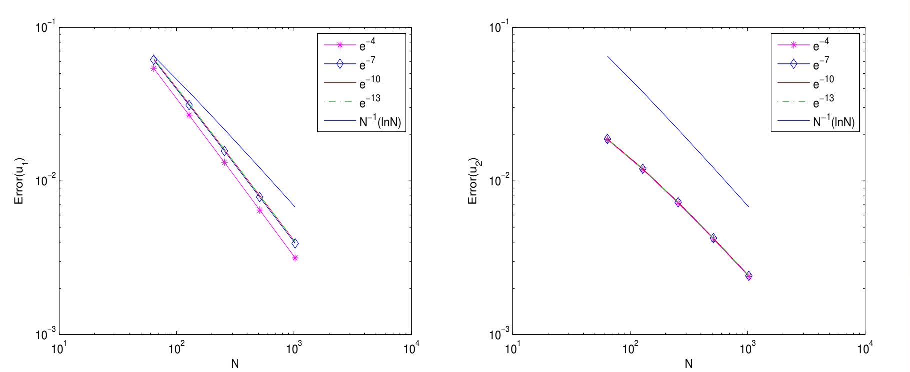

Figure: Maximum pointwise errors as a function of N and ε for the solution u_1 and u_2 for Example 26

Conclusion and Scope

- Finite Difference method and Variable mesh spline approximation method are applied for problem class I. Finite Difference method is used solve the remaining problems.
- One can apply these method for other class of problems like multiple turning point problems, turning point problem with interior layers, two parameter turning point problems, turning point problem with discontinuous source term, etc.

List of Publications

(1) N. Geetha, A. Tamilselvan, V. Subburayan, *Parameter uniform numerical method for third order singularly perturbed turning point problems exhibiting boundary layers,* Int. J. Appl. Comput. Math., DOI 10.1007/s40819-015-0064-4. (2015) **(Springer)**

(2) N. Geetha, A. Tamilselvan, *Numerical method for system of second order singularly perturbed turning point problems with Robin boundary conditions,* Procedia Engineering, 127 (2015), 670 â 677. **(Elsevier)**

(3) N. Geetha, A. Tamilselvan, *Variable mesh spline approximation method for solving second order singularly perturbed turning point problems with Robin boundary conditions,* International Journal of Applied and Computational Mathematics, DOI 10.1007/s40819-015-0064-4. (2016). **(Springer)**

(4) N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for fourth order singularly perturbed turning point problems exhibiting boundary layers*, Ain Shams Engineering Journal (2016), http://dx.doi.org/10.1016/j.asej.2016.04.018. **(Elsevier)**

(5) N. Geetha, A. Tamilselvan, *Numerical method for system of second order singularly perturbed turning point problems exhibiting boundary layers,* Journal of Mathematical Modeling, Vol. 6, No. 2, 2016, pp. 211-232.

List of articles communicated for publication

1. N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for second order singularly perturbed turning point problems with Robin boundary conditions*.
2. J. Christy Roja, A. Tamilselvan, N. Geetha, *A parameter- uniform second order Schwarz method for a weakly coupled system of singularly perturbed convection diffusion equations.*

Problems working on

- N. Geetha, A. Tamilselvan, *Numerical method for a two parameter singularly perturbed turning point problems*.
- N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for singularly perturbed turning point problems with discontinuous source term*.

References

[1] L. R. Abrahamsson, *A priori estimates for solutions of singular perturbations with a turning point*, Studies in applied mathematics **56**, 51-69 (1977).

[2] R. C. Ackerberg and R. E. O'Malley ,*Boundary layer problems exibiting resonance*, Stdies in aplied mathematics **XLIX (30)**, 277-295 (1970).

[3] C. M. Bender, S. A. Orzag, *Advanced Mathematical methods for Scientist and Engineers*, McGrawhill. New York, (1978).

[4] A. E. Berger, H. Han and R. B. Kellogg, *A priori estimates and analysis of a numerical method for a turning point problem*, Mathematics of computation **42**, 465-492,(1984).

[5] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole press, Dublin(1980).

[6] F. A. Howes, *An asymptotic-numerical method for a model system with turning points*, Journal of Computational and Applied Mathematics, **29**, 343-356, (1990).

[7] T. C. Hanks, Model relating heat flow values near, and verticle velocities of mass transport beneath, ocean rices, Geohys. Res. **76**, 537-544 (1971).

[8] H. Schlichting, Boundary-layer theory, McGRaw-hill, New York, (1979.)

[9] Kapil K. sharma, Pratima rai, Kailash C. Patidar, *A review on singularly perturbed differential equations with turning points and interiror layers*, Applied Mathematics and computation, **219**, 10575-10609, (2013).

[10] R. E. O' Malley, *Introduction to singular perturbations*, Academic Press, New York, (1974).

[11] Anthony Leung, *A third order linear differential equation on the real line with two turning points,* Journal of differential equations **29**, 304-328, (1978).

[12] H. G. Roos, M. Stynes, L. Tobiska, *Numerical methods for singularly perturbed differential equations-convection-diffusion and flow problems*, Springer, (2006).

[13] W. Wasow, *Linear turning point theory*, Springer Verlag, New York, (1984).

[14] A. M. Watts, *A Singular perturbation problem with a turning point*, Bull. Austral. Math. Soc, Vol **5**, 61-73, (1971).

[15] P. A. Ferrell, *Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem*, SIAM J. on Numerical Analysis **25**(3), 618-643 (1988).

[16] M. K. Kadalbajoo, K. C. Patidar, *Variable mesh spline approximation method for solving singularly perturbed turning point*

problems having boundary layer(s), Computer Mathematics with Applications, Vol **42**, 1439- 1453, (2001).

[17] A. Tamilselvan, N. Ramanujam, R. Mythili Priyadharshini and T. Valanarasu *Parameter uniform numerical method a system of coupled singularly perturbed convection diffusion equations with mixed type boundary conditions*, J. Appl. Math. & Informatics, Vol **28**, 109-130,(2010)

[18] S. Natesan, N. Ramanujam, *A computational method for solving singularly perturbed turning point problems exhibiting twin boundary layers*, Applied Mathematics and computation, Vol **93**, 259-275, (1998).

[19] S. Natesan, N. Ramanujam, *Initial-Value Technique for Singularly Perturbed Turning Point Problems Exhibiting Twin Boundary Layers*, Journal of Optimization Theory and Applications, **99**(1), 37-52, (1998).

[20] S. Natesan, J. Jayakumar, J. Vigo-Aguiar, *Parameter uniform numerical method for singularly perturbed turning point problems*

exhibiting boundary layers, Journal computational and applied Mathematics, vol **158**, 121-134, (2003)

[21] Jia-qi Mo, Zhao-hui Wen, *A class of boundary value problems for third order differential equation with a turning point*, Applied Mathematics and Mechnics **31**(8), 1027-1032, (2010)

[22] F.Z. Geng, S.P. Qian, S. Li., *A numerical method for singularly perturbed turning point problems with an interior layer*, Journal of Computational and Applied Mathematics **255**, 97-105, 2014.

Thank You