WELCOME

On Colouring of Digraphs

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Colouring Of Digraphs

A colouring of a digraph $D = (V, A)$ is a colouring of its vertices by the following rule: Let uv be an arc in D. If the tail u is coloured first, then the head v should receive a colour different from that of u. But, if v is coloured first, then u may or may not receive the colour of v. This type of colouring for digraph was defined by E. Sampathkumar.

- The dichromatic number $\chi_d(D)$ of a digraph D is the minimum number of colours needed for colouring of D.
- If $\chi_d(D) = k$, then D is said to be k-dichromatic.
- For a digraph D, the *underlying graph* of D, denoted by $G(D)$ has the same vertex set as that of D, where two vertices u and v are adjacent if uv or vu is an arc in D.
- A proper colouring of a graph G is a colouring of its vertices such that no two adjacent vertices receive the same colour.
- The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed in a proper colouring of G.

- Let $C = \{c_1, c_2, ..., c_k\}$ be a colouring of D and $V(D) = \{v_1, v_2, ..., v_n\}$. Corresponding to C, there exists a sequence of vertices $(v_1, c_1), (v_2, c_2), ..., (v_n, c_n)$ such that for $i < j$, $c_i \neq c_j$ if $v_i v_j$ is an arc. This sequence is called *colour sequence.*

PRELIMINARIES

- A directed walk in D is a finite alternating sequence $W = v_0 a_1 v_1 a_2 v_2 ... a_k v_k$ in which a_i is an arc joining v_{i-1} with v_i, for $i = 1,2,3,...,k$.
- A directed path is a directed walk $v_0 a_1 v_1 a_2 v_2 ... a_k v_k$, in which all $v_i^{,}s$ are distinct.
- A directed walk $v_0 a_1 v_1 a_2 v_2 ... a_k v_k$ is a directed cycle if all $v_i^{,}s$ are distinct except v_0 and v_k and $v_0 = v_k$.
- A digraph D is a tree if its underlying graph $G(D)$ is a tree.
- A digraph D is a tournament if there is an arc between every pair of vertices.
- A digraph D is acyclic if it does not contain a directed cycle.
- The complement of a digraph D, denoted by $\overline{D}$, is defined by taking $V(\overline{D}) = V(D)$ and for every ordered pair (u, v), uv is an arc of $\overline{D}$ if and only if, uv is not an arc of D.

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Definition 1: Let T be a tournament. Replace each arc in T by a pair of symmetric arcs. The digraph thus obtained is a symmetric complete graph and is denoted by K_n^*.

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RESULTS

Here, we have some results given by E. Sampathkumar in [4] which will be used to prove our main results.

Lemma

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Observation: For any digraph D, $\chi_d(D) \leq \chi(G(D))$, where $\chi(G(D))$ is the chromatic number of the graph $G(D)$.

- Given any two positive integers a and b, $a \leq b$, it is possible to construct a digraph D such that $\chi_d(D) = a$ and $\chi(G(D)) = b$.
- Take an acyclic tournament T on b vertices. Choose any a vertices from T and add arcs between these a vertices so that these a vertices form K_a^*.
- The digraph D thus obtained has dichromaticity a and the corresponding underlying graph $G(D)$ has chromaticity b.

Theorem

For a digraph D without symmetric arcs, $\chi_d(D) = 1$ or 2.

Proof.

- If D is acyclic, then by lemma 4, $\chi_d(D) = 1$.
- Let D have one or more directed cycles. By removing at most one arc from each directed cycle, we get an acyclic subdigraph D^0 such that $\chi_d(D^0) = 1$.
- Let the arcs that are removed be $u_1v_1, u_2v_2, \ldots, u_kv_k$.
- Let C be a colour sequence for D^0. In this sequence each v_i precedes u_i. In D^0, all the vertices are assigned the same colour say c_1. Let $D' = D^0 + u_1v_1$.

∎

Proof.

- By changing the colour of v_1 to c_2 in $D^{'}$, we get a colouring for $D^{'}$ such that $\chi_d(D^{'}) = 2$.
- Let $D^{''} = D^{'} + u_2 v_2$. In D'', we can change the colour of v_2 to c_2 from c_1, if u_2 and v_2 have the same colour. Hence $\chi_d(D^{''}) = 2$.
- Repeating the above argument till we get D and $\chi_d(D) = 2$.

∎

Corollary

For any tournament T, $\chi_d(T) \leq 2$.

In [4], it is given that $\chi_d(D) = 2$ if and only if D does not have an odd symmetric cycles. This is false. We have a family of digraphs without odd symmetric cycles with dichromaticity 3. One such example is shown in figure 1.

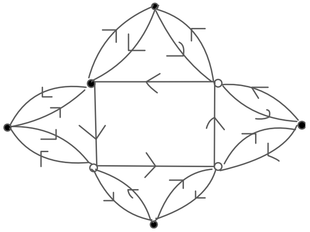

Figure: D

Next, we give necessary and sufficient condition for $\chi_d(D) = 2$.

Theorem

Let D be a digraph. Then $\chi_d(D) = 2$ if and only if either D has no symmetric cycle but has at least one directed cycle or D has only even symmetric cycles and the set of vertices in the odd or even positions in each symmetric cycle do not form a directed cycle.

Proof.

- If D has no symmetric cycle and has at least one directed cycle, then by theorem 1, $\chi_d(D) = 2$.
- Let D have only even symmetric cycle and the set of vertices in the odd or even positions in the symmetric cycle do not form a directed cycle.

∎

Proof.

- Remove one arc from each symmetric cycle of D so that the resultant digraph D' has no symmetric cycle. Hence $\chi_d(D') = 2$.
- In any 2-colouring of D', the two end vertices of the removed arcs receive distinct colours so that $\chi_d(D) = 2$.
- Conversely, let $\chi_d(D) = 2$. Obviously D has a directed cycle and no odd symmetric cycles. If D has no even symmetric cycle, then it must have at least one directed cycle, since $\chi_d(D) = 2$.

∎

Proof.

- Suppose D has an even symmetric cycle. In case the set of vertices in the odd and even positions of the symmetric cycle form a directed cycle, then $\chi_d(D) = 3$. Hence the result follows.

∎

Theorem

For any digraph D on n vertices, $\chi_d(D) = n$ if and only if $D \cong K_n^*$

Proof.

- Let $\chi_d(D) = n$.
- Suppose $D \ncong K_n^*$, then there exists atleast two vertices say u and v such that either there is no arc between u and v or there is only one arc between u and v.
- In both the cases, we can always obtain a colouring in which both u and v receive the same colour. So $\chi_d(D) < n$, a contradiction. Hence $D \cong K_n^*$.
- Converse is obvious.

■

Theorem

Let D be a digraph on $n \geq 3$ vertices. Then $\chi_d(D) = n - 1$ if and only if $D \in (\mathscr{G} \cup \mathscr{D})$.

Proof.

- Let $\chi_d(D) = n - 1$. Obviously $D \neq K_n^*$.
- Then there exists two vertices u and v such that there is no symmetric arc between them.
- If there is no arc between u and v, then D contains K_{n-1}^*, for otherwise, $\chi_d(D) \leq n - 2$.In this case, $D \in \mathscr{G}$.
- Let us assume that D has an arc uv.

∎

Proof.

- We claim that D has K^*_{n-2} as a subdigraph of D and u and v does not belong to this K^*_{n-2}.
- Suppose not, then we can always find $(n-2)$ vertices which need at most $(n-3)$ colours resulting that $\chi_d(D) \leq n-2$, a contradiction.

∎

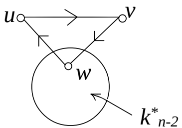

Figure: D_1

Proof.

- Obviously u and v does not belong to this K^*_{n-2}.
- If both u and v are in symmetric with every vertices of K^*_{n-2}, then $D \in \mathscr{G}$.
- Let us assume the case where u or v is not symmetric with atleast one vertex of K^*_{n-2}.
- If there exist two or more such vertices in K^*_{n-2}, then $\chi_d(D)$ $\leq n-2$. So there exist exactly one vertex say w in K^*_{n-2} to which either u or v is not symmetric.

∎

Proof.

- Suppose only u is not symmetric with w, then $D \in \mathscr{G}$.
- If both u and v are not symmetric with w and uvw is not a directed cycle, then $\chi_d(D) \leq n - 2$. This case never occurs.
- If uvw forms a directed cycle, then $D \in \mathscr{D}$. Hence the theorem.

∎

In [4], it is mentioned that $\chi_d(D) = \chi(G(D))$ if and only if every arc in D is symmetric. We have a counter example. A directed even cycle disproves this statement.

Theorem

Let D be a digraph. $\chi_d(D) = \chi(G(D)) = 2$ if and only if D has a directed cycle but none is odd .

Proof.

- Let $\chi_d(D) = \chi(G(D)) = 2$. Since $\chi_d(D) = 2$, by lemma 1, D has at least one directed cycle.
- This cycle cannot be odd, for otherwise, $G(D)$ will have an odd cycle, contradicting the fact $\chi(G(D)) = 2$.

∎

Proof.

- Conversely, let D have a directed cycle but none is odd. Therefore $\chi_d(D) \geq 2$.
- Now $G(D)$ has no odd cycle. So $\chi(G(D)) = 2$. Hence $2 \leq \chi_d(D)) \leq \chi(G(D)) = 2$. This implies $\chi_d(D) = \chi(G(D)) = 2$.

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OBSERVATION: $\chi_d(D) = \chi(G(D)) = n$ if and only if D is K_n^*.

NORDHAUS-GADDUM TYPE RESULTS FOR DIGRAPH

Bounds on the sum and product of $\chi_d(D)$ and $\chi_d(\overline{D})$ can be obtained by the following theorem.

Theorem

Let D be a digraph without symmetric arcs. Then

$$2 \leq \chi_d(D) + \chi_d(\overline{D}) \leq n + 1$$

and

$$1 \leq \chi_d(D)\chi_d(\overline{D}) \leq 2(n - 1)$$

with left equalities hold if and only if D is an acyclic tournament and right equalities if and only if, D is a directed cycle on three vertices together with (n-3) isolated vertices.

Proof.

- If D is a digraph without symmetric arcs, then $\chi_d(D) \leq 2$ and $\chi_d(\overline{D}) < \chi(G(\overline{D}))= \chi(K_n) = n$.
- Therefore $\chi_d(D) + \chi_d(\overline{D}) \leq 2 + (n-1) = n+1$ and $\chi_d(D)\chi_d(\overline{D}) \leq 2(n-1)$.
- Now, let $\chi_d(D) + \chi_d(\overline{D}) = n+1$. Since $\chi_d(\overline{D}) \leq n-1$ and $\chi_d(D) \leq 2$, we have $\chi_d(\overline{D}) = n-1$.
- By theorem 4, $\overline{D} \in (\mathscr{G} \cup \mathscr{D})$.
- In case $\overline{D} \in \mathscr{G}$, either D has symmetric arcs or $\chi_d(D) = 1$, both cannot occur.

∎

Proof.

- Therefore $\overline{D} \in \mathscr{D}$ and hence D is a directed cycle on three vertices together with $n-3$ isolated vertices.
- Now, let $\chi_d(D) + \chi_d(\overline{D}) = 2$. This implies that $\chi_d(D) = \chi_d \overline{D} = 1$.
- By converse of the lemma 4, D and $\overline{D}$ are acyclic.
- Therefore, between every pair of vertices in D and $\overline{D}$ there is only one arc. Hence D and $\overline{D}$ are acyclic tournaments.
- Similarly we can prove the other equalities.

∎

Corollary

If T is a tournament with $\chi_d(T) = 1$, then $\chi_d(T) + \chi_d(\overline{T}) = 2$ and $\chi_d(T)\chi_d(\overline{T}) = 1$.

Proof.

- Since $\chi_d(T) = 1$, T has no directed cycle.
- Obviously $\overline{T}$ also has no directed cycle and $\chi_d(\overline{T}) = 1$. Hence the proof.

∎

Corollary

If T is a tournament with $\chi_d(T) = 2$, then $\chi_d(T) + \chi_d(\overline{T}) = \chi_d(T)\chi_d(\overline{T}) = 4$.

Proof.

- Since $\chi_d(T) = 2$, T has a directed cycle.
- Obviously $\overline{T}$ also has a directed cycle and $\chi_d(\overline{T}) = 2$. This completes the proof.

∎

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THANK YOU